Pole-Placement Design – A Polynomial Approach

Overview

- A Simple Design Problem
- The Diophantine Equation
- More Realistic Assumptions
A Simple Design Problem

• System:

\[ A(q)y[k] = B(q)u[k] \]  \hspace{1cm} (1)

• Assumptions:
  
  – \( A = q^{n_a} + a_1 q^{n_a-1} + \cdots + a_{n_a-1} q + a_{n_a} \) is monic
  
  – \( \text{deg } A > \text{deg } B \).

  – Disturbances are widely spread impulses

• Specifications are given by the closed-loop characteristic polynomial, and the controller may have certain properties, for example, integral action

• General linear controller (\( u_c \): command signal, \( y \): measured output, \( u \): control signal):

\[ R(q)u[k] = T(q)u_c[k] - S(q)y[k] \]  \hspace{1cm} (2)
• $R(q)$ is chosen to be monic.

• Feedforward part:

\[ H_{ff}(z) = \frac{T(z)}{R(z)} \quad (3) \]

• Feedback part:

\[ H_{fb}(z) = \frac{S(z)}{R(z)} \quad (4) \]

• Goal: Causal controller with no time-delay ⇒

\[ \deg R = \deg S \quad (5) \]
Solving the design process

• Eliminating $u[k]$ between the process model (1) and the controller (2) gives

\[
(A(q)R(q) + B(q)S(q)) y[k] = B(q)T(q)u_c[k]
\]  

(6)

• Closed-loop characteristic polynomial:

\[
A_{cl}(z) = A(z)R(z) + B(z)S(z)
\]  

(7)

• The pole-placement design is to find polynomials $S$ and $R$ that satisfy Eq. (7) for given $A$, $B$, and $A_{cl}$.

• Eq. (7) is called Diophantine Equation.

• Factorising the $A_{cl}$ polynomial:

\[
A_{cl}(z) = A_c(z)A_o(z)
\]  

(8)
• We call $A_c(z)$ the controller polynomial and $A_o(z)$ the observer polynomial.

• In order to determine the polynomial $T$, we calculate the pulse-transfer function from the command signal to the output:

$$Y(z) = \frac{B(z)T(z)}{A_{cl}(z)}U_c(z) = \frac{B(z)T(z)}{A_c(z)A_o(z)}U_c(z)$$  \hspace{1cm} (9)

• Zeros of the open-loop system are also zeros of the closed-loop system (unless $B(z)$ and $A_{cl}(z)$ have common factors).

• Let’s choose the polynomial $T$ so that it cancels the observer polynomial $A_o$:

$$T(z) = t_0A_o(z)$$  \hspace{1cm} (10)

• The response to command signals is then given by

$$Y(z) = \frac{t_0B(z)}{A_c(z)}U_c(z)$$  \hspace{1cm} (11)

where $t_0$ is chosen to obtain a desired static gain for the system (e.g., for unit gain: $t_0 = A_c(1)/B(1)$).
The Diophantine Equation - Minimal-Degree Solution:

- The equation

\[ A_{cl}(z) = A(z)R(z) + B(z)S(z) \]  \hspace{1cm} (12)

has a solution only if the greatest common divisor of \( A \) and \( B \) divides \( A_{cl} \).

- Number of controller parameters when \( \text{deg} \ R = \text{deg} \ S \):

\[ n_p = 2(\text{deg} R + 1) = 2 \text{deg} R + 2 \]  \hspace{1cm} (13)

- Degree of \( A_{cl} \) (remind that \( \text{deg}(AR) > \text{deg}(BS) \)):

\[ \text{deg}(A_{cl}) = \max(\text{deg}(AR), \text{deg}(BS)) = \text{deg}(AR) = \text{deg}(A) + \text{deg}(R) \]  \hspace{1cm} (14)

- Number of equations (coefficients of \( A_{cl} \)):

\[ n_e = \text{deg}(A_{cl}) + 1 = \text{deg}(A) + \text{deg}(R) + 1 \]  \hspace{1cm} (15)
• Unique minimal solution:

\[ n_e = n_p \]  \hfill (16)

\[ \deg A + \deg R + 1 = 2 \deg R + 2 \]  \hfill (17)

\[ \Rightarrow \]

\[ \deg R = \deg A - 1 \]  \hfill (18)

• Degree of the closed-loop polynomial for the minimum-degree solution:

\[ \deg A_{cl} = 2 \deg(A) - 1 \]  \hfill (19)
• The Diophantine equation can be solved using matrix calculations \((n = n_a > n_b)\):

\[
\begin{pmatrix}
a_0 & 0 & 0 & \cdots & 0 & b_0 & 0 & 0 & \cdots & 0 \\
\bar{a}_1 & a_0 & 0 & \cdots & 0 & b_1 & b_0 & 0 & \cdots & 0 \\
\bar{a}_2 & \bar{a}_1 & a_0 & \cdots & 0 & b_2 & b_1 & b_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & a_0 & b_n & b_{n-1} & b_{n-2} & \cdots & b_0 \\
0 & \bar{a}_n & \bar{a}_{n-1} & \cdots & a_1 & b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \\
0 & 0 & \bar{a}_n & \cdots & \bar{a}_2 & 0 & 0 & b_n & \cdots & b_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \bar{a}_n & 0 & 0 & 0 & 0 & \cdots & b_n \\
\end{pmatrix} \begin{pmatrix}
r_0 \\
r_1 \\
r_{n_a-1} \\
s_0 \\
s_{n_a-1} \\
\end{pmatrix} = \begin{pmatrix}
a_{cl,0} \\
a_{cl,1} \\
a_{cl,2} \\
a_{cl,n} \\
a_{cl,n+1} \\
a_{cl,n+2} \\
\end{pmatrix}
\]

\[
\bar{B} = b_0 q^n + b_1 q^{n-1} + \cdots + b_{n-1} q + b_n = b_0 q^{n_b} + b_1 q^{n_b-1} + \cdots + b_{n_b-1} q + b_{n_b} \quad (21)
\]

\[
\bar{A} = a_0 q^n + a_1 q^{n-1} + \cdots + a_{n-1} q + a_n = 1 q^{n_a} + a_1 q^{n_a-1} + \cdots + a_{n_a-1} q + a_{n_a} \quad (22)
\]
More Realistic Assumptions

Cancellations of Poles and Zeros

- Factorisation of $A$ and $B$:

$$A = A^+ A^-$$
$$B = B^+ B^-$$  \hspace{1cm} (23)  \hspace{1cm} (24)

- $A^+$ and $B^+$ are stable (well damped) factors that can be cancelled (both should be chosen monic).

- Poles that shall be cancelled must be controller zeros and zeros that must be cancelled must be controller poles:

$$R = B^+ R_d \bar{R}$$
$$S = A^+ S_d \bar{S}$$  \hspace{1cm} (25)  \hspace{1cm} (26)

where $R_d$ and $S_d$ are fixed pre-determined parts of the controller (see next subsection).
• Closed-loop polynomial:

\[ A_{cl} = AR + BS = A^+ B^+ (R_d \overline{RA}^- + S_d \overline{SB}^-) = A^+ B^+ \overline{A}_{cl} \] (27)

• Cancelled zeros and poles are part of the closed-loop polynomial and must therefore be well damped!

• Cancelling the common factors we find that the polynomials \( \overline{R} \) and \( \overline{S} \) must satisfy:

\[ \overline{RR}_d A^- + \overline{SS}_d B^- = \overline{A}_{cl} \] (28)

• Minimal-degree solution:

\[ n_e = n_p \] (29)

\[ \deg \overline{A}_{cl} + 1 = \deg \overline{R} + \deg \overline{S} + 2 \] (30)

\[ \max(\deg \overline{RR}_d A^-, \deg \overline{SS}_d B^-) + 1 = \deg \overline{R} + \deg \overline{S} + 2 \] (31)

• For \( \deg \overline{RR}_d A^- > \deg \overline{SS}_d B^- \) we obtain:

\[ \deg \overline{S} = \deg A^- + \deg R_d - 1 \] (32)

\[ \deg(\overline{RR}_d A^-) = \max(\deg \overline{A}_{cl}, \deg \overline{SS}_d B^-) = \deg \overline{A}_{cl} \] (33)
\[ \text{deg } S = \text{deg } R \quad (34) \]

\[ \begin{align*}
\text{deg } A^- + \text{deg } R_d - 1 + \text{deg } S_d + \text{deg } A^+ & = \\
\text{deg } S & = \\
\text{deg } \bar{A}_{cl} - \text{deg } A^- - \text{deg } R_d + \text{deg } R_d + \text{deg } B^+ & = \\
\text{deg } \bar{R} & = \\
\end{align*} \quad (35) \]

Solving this equation for \( \text{deg } \bar{A}_{cl} \) yields:

\[ \text{deg } \bar{A}_{cl} = 2 \text{deg } A + \text{deg } R_d + \text{deg } S_d - \text{deg } B^+ - \text{deg } A^+ - 1 \quad (36) \]

\[ \text{deg } A_{cl} = 2 \text{deg } A + \text{deg } R_d + \text{deg } S_d - 1 \quad (37) \]
• In order to solve Eq. (28) we create polynomials $\overline{A}$ and $\overline{B}$ with $\deg \overline{A} = \deg \overline{B} = n = \max(\deg(A - R_d), \deg(B - S_d))$:

$$\overline{A} = A^{-R_d}$$
$$\overline{B} = B^{-S_d}$$

\[ \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & a_{\deg \overline{A}} & a_{\deg \overline{A} - 1} & \ddots & \ddots \\
\ddots & 0 & a_{\deg \overline{A}} & \ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
\overline{r}_0 \\
\ddots \\
\overline{s}_0 \\
\end{pmatrix}
= \begin{pmatrix}
a_{cl0} \\
\ddots \\
a_{cl1} \\
\end{pmatrix} \tag{40} \]
Handling disturbances

\[ x = \frac{BT}{AR + SB} u_c + \frac{BR}{AR + SB} v - \frac{BS}{AR + SB} e \]

\[ y = \frac{BT}{AR + SB} u_c + \frac{BR}{AR + SB} v + \frac{AR}{AR + SB} e \]

\[ u = \frac{AT}{AR + SB} u_c - \frac{BS}{AR + SB} v - \frac{AS}{AR + SB} e \]
• To avoid steady-state errors due to constant load disturbances the static gain from the disturbance $v$ to $y$ must be zero:

$$B(1)R(1) = 0$$

If $B(1) \neq 0$ then we must require that $R(1) = 0$. This means that $R_d = z - 1$ is a factor of $R(z)$ or that the controller is required to have integral action.

• Elimination of periodic load disturbances (with period $n \cdot \Delta$) by using $R_d = z^n - 1$:

$$v((k + n)\Delta) - v(k\Delta) = (q^n - 1)v(k\Delta) = 0$$

• Elimination of sinusoidal load disturbances with frequency $\omega_0$:

$$R_d = z^{-2} + z \cos(\omega_0 \Delta) + 1$$

• Eliminating the effect of measurement noise at Nyquist frequency:

$$S_d = z + 1$$
Pre-filter revisited

- Let us factorise the polynomial $A_{cl} = \frac{A^+ A_o B^+ A_c}{A_o A_c}$

- Let’s choose the polynomial $T$ so that is cancels the observer polynomial $A_o$:

$$T(z) = t_0 A_o(z)$$  \hspace{1cm} (41)

- The response to command signals is then given by

$$Y(z) = \frac{t_0 B(z)}{A_c(z)} U_c(z) = H_m(z) U_c(z)$$  \hspace{1cm} (42)

where $t_0$ is chosen to obtain a desired static gain for the system (e.g., for unit gain: $t_0 = A_c(1)/B(1)$).

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