Discrete-Time Systems

Overview

- Sampling a Continuous-Time State-Space Model
- Input-Output Models: The Pulse Response
- Shift Operator
- The z-Transform
- Computation of the Pulse-Transfer Function
- Poles and Zeros

Sampling a Continuous-Time State-Space Model

- Assuming a continuous-time system given in the following state-space form

\[
\begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

The system has \( r \) inputs, \( p \) outputs, and is of order \( n \).

- General solution \( x(t) \) of the state over a time interval starting at \( t_0 \),

\[
x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds
\]

- Assuming the time interval now goes from one sampling instant \( t_0 = k\Delta + \Delta \) to the next sampling instant \( t = (k+1)\Delta + \Delta \).

- The control signal \( u(t) \) is kept constant over the sampling interval:

\[
u(s) = u(k\Delta), k\Delta \leq s < (k+1)\Delta
\]

Using the definition of the matrix exponential yields special structures.

Calculation by hand are feasible for low order systems, \( n \leq 2 \), and for high-order systems with special structures.

Using the definition of the matrix exponential yields

\[
\Phi(\Delta) = e^{A\Delta} = I + A\Delta + \frac{A^2\Delta^2}{2!} + \cdots + \frac{A^n\Delta^n}{n!} + \cdots
\]

which can be rewritten into

\[
\Phi(\Delta) = I + A\Phi(\Delta)
\]

Solution of the System Equation

Assume that the initial condition \( x[k_0] \) and the input signals \( w[k_0], w[k_0+1], \ldots \) for the discrete-time system in state-space form are given.

\[
x[k_0 + 1] = \Phi x[k_0] + \Gamma w[k_0]
\]

\[
x[k_0 + 2] = \Phi x[k_0 + 1] + \Gamma w[k_0 + 1]
\]

\[
\vdots
\]

\[
x[k] = \Phi^{k-k_0} x[k_0] + \Phi^{k-k_0-1} \Gamma w[k_0] + \cdots + \Gamma w[k - 1]
\]

The solution consists of two parts: One depends on the initial condition and the other is the weighted sum of the input signals.
The solution is now simple. Each mode will have the solution:

\[ z_i[k] = L_k \] 

where \( L_k \) is a \( k \times k \) matrix.

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The characteristic equation

\[ \det(\lambda I - \Phi) = 0 \]

is invariant when new states are introduced through the nonsingular transformation matrix \( T \).

Proof:

\[ \det(\lambda ITT^{-1} - T\Phi T^{-1}) = \det(T) \det(\lambda I - \Phi) \det T^{-1} = \det(\lambda I - \Phi) \]

To find a transformation it is the same as solving for the \( n^2 \) elements of \( T \) from a linear set of equations:

\[ T\Phi - \Phi T - (T\Phi T^{-1}) = 0 \]

Coordinates can be chosen to give simple forms of the system equations.

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Changing Coordinates in State-Space Models

Assume \( \Phi \) has distinct eigenvalues. Then there exists a \( T \) such that

\[ T\Phi T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots \\ 0 & 0 & \lambda_n \end{pmatrix} \]

where \( \lambda_i \) are the eigenvalues of \( \Phi \). Assumeing single input \( u[k] \), with the transformation a set of decoupled first order difference equations is obtained:

\[ z_i[k+1] = \lambda_i z_i[k] + \beta_i u[k] \]

\[ y[k] = \gamma_1 z_1[k] + \cdots + \gamma_n z_n[k] \]

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Diagonal Form

The characteristic equation

\[ \det(\lambda I - \Phi) = 0 \]

will determine the properties of the solution.

The Eigenvalues are obtained from the characteristic equation:

\[ \det(\lambda I - \Phi) = 0 \]

Example (blackboard) ...

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Jordan Form

If \( \Phi \) has multiple Eigenvalues, then it is generally not possible to diagonalize \( \Phi \). Introduce the notation

\[ L_\lambda(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix} \]

where \( L_\lambda \) is a \( k \times k \) matrix.
Consider the following higher-order difference equation
\[ y[k + n_a] + a_1 y[k + n_{a-1}] + \ldots + a_{n_a} y[k] = b_0 u[k + n_d] + \ldots + b_{n_a} u[k] \]

A general linear model that relates input and output can also be written as
\[ y[k] = a_0 y_{p}[k] + \sum_{m=0}^{k} h(k,m) u[m] \]

where \( y_{p} \) accounts for initial conditions of the system.

- The function \( \Pi(k, m) \) is called the pulse-response function.
- The pulse-response function can be easily measured by injecting a pulse of unit magnitude and the width of the sampling interval and recording the output. For zero initial conditions the value \( \Pi(k, m) \) is the output at sampling instant \( k \) for a unit pulse at instant \( m \).
- For time-invariant systems the pulse response becomes
  \[ \Pi(k, m) = h(k - m) \]

Attention:
- It is easy to compute the pulse response of the system defined by the state-space model 
  \( k_0 = 0 \):
  \[ y[k] = C\Phi^{k-k_0} x[0] + \sum_{j=0}^{k-k_0} C\Phi^{k-j-1} \Gamma u[j] + D_{y}[k] \]
  \[ h(k) = \begin{cases} 0 & k < 0 \\ D & k = 0 \\ C\Phi^{k-1} \Gamma & k \geq 1 \end{cases} \]
- The pulse response is invariant with respect to the coordinate transf. of the state-space model.

Consider the following higher-order difference equation
\[ y[k + n_a] + a_1 y[k + n_{a-1}] + \ldots + a_{n_a} y[k] = b_0 u[k + n_d] + \ldots + b_{n_a} u[k] \]

where \( n_a \) is the order of the difference equation, \( n_u \) is the number of inputs, and \( n_y \) is the number of outputs.

The before introduced higher-order difference equation can be expressed in terms of back-shift operator
\[ y[k] + a_1 y[k-1] + \ldots + a_{n_a} y[k-n_{a}] = b_0 u[k-d] + \ldots + b_{n_a} u[k-d-n_u] \]

where \( d = n_u - n_a \) is the pole excess (time-delay) of the system.

The reciprocal polynomial
\[ A^*(q) = 1 + a_1 q + \ldots + a_{n_a} q^{n_a} = q^{-n} A(q^{-1}) \]

is obtained from the polynomial \( A \) by reversion of the order of the coefficients. With reciprocal polynomials the system can we write
\[ A^*(q^{-1}) y[k] = B^*(q^{-1}) u[k-d] \]

Attention: \( A^{-n} \) is not necessary the same as \( A \) (see for example \( A(q) = q \)). A polynomial is self-reciprocal if
\[ A^*(q) = A(q) \]
A difficulty with the shift operator

- Difference equations can be multiplied by powers of $q$.
- Equations for shifted times can be multiplied by real numbers and added (corresponds to multiplying by a polynomial in $q$).
- However, it is not possible to divide by a polynomial in $q$ unless special assumptions are made.
- Division by an arbitrary polynomial is only allowed if it is assumed that there is some $k_0$ such that all sequences are zero for $k \leq k_0$.

Example (see blackboard) ...

The pulse transfer-operator is thus given by

$$H(q) = \frac{B(q)}{A(q)}$$

The pulse transfer-operator $H(q)$ for a state-space model is independent of the state representation.

The pulse transfer-operator can also be expressed in terms of the backward-shift operator

$$(qI - \Phi)$$

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Example (see blackboard) ...

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- The discrete-time analogy of the Laplace transform is the $z$-transform – a convenient tool to study linear difference equations without and with initial conditions (in contrast to the calculus operator).
- The $z$-Transform maps a semi-infinite time sequence (in contrast to the operator calculus) into a function of a complex variable.
- The $z$-Transform is defined as

$$Z\{f(k\Delta)\} = F(z) = \sum_{k=0}^{\infty} f(k\Delta)z^{-k}$$

where $z$ is a complex variable (in contrast to $q$).

- The inverse transform is given by

$$f(k\Delta) = \frac{1}{2\pi j} \int F(z)z^{k-1}dz$$

Some properties of the $z$-Transform ($h = \Delta$):

1. Defining
   $$F(z) = \sum_{k=0}^{\infty} f(k\Delta)z^{-k}$$
   $$f(k\Delta) = \frac{1}{2\pi j} \int F(z)z^{k-1}dz$$

2. Inversion
   $$f(k\Delta) = \frac{1}{2\pi j} \int F(z)z^{k-1}dz$$
   $$F(z) = \sum_{k=0}^{\infty} f(k\Delta)z^{-k}$$

3. Linearity
   $$Z\{f+g\} = Z\{f\} + Z\{g\}$$

4. Time shift
   $$Z\{f(k\Delta - r)\} = z^{-r}F(z)$$
   $$Z\{f(k\Delta - r)\} = \sum_{k=0}^{\infty} f(k\Delta)z^{-k}$$

5. Initial-value theorem
   $$f(0) = \lim_{z \to 1} (z-1)F(z)$$

6. Final-value theorem
   $$\lim_{z \to 0} (1-z)F(z) = \lim_{k \to \infty} f(k\Delta)$$

7. Convolution
   $$Z\{f\ast g\} = Z\{\sum_{n=0}^{\infty} f(n\Delta)g(k-n)\} = (Zf)(Zg)$$
The z-Transform can be used to solve difference equations, for instance
\[ x[k + 1] = \Phi x[k] + \Gamma u[k] \]
\[ y[k] = Cx[k] + D_0 u[k] \]

If the z-transform of both sides is taken
\[ \sum_{k=0}^{\infty} z^{-k} x[k + 1] = z \left( \sum_{k=0}^{\infty} z^{-k} x[k] \right) - \sum_{k=0}^{\infty} \Phi z^{-k} x[k] + \sum_{k=0}^{\infty} \Gamma z^{-k} u[k] \]

Hence
\[ z \{ X(z) - x[0] \} = \Phi X(z) + \Gamma U(z) \]
\[ X(z) = \frac{(zI - \Phi)^{-1} x[0] + \Gamma U(z)}{1 - \Phi} \]
\[ Y(z) = C(zI - \Phi)^{-1} x[0] + \frac{C(zI - \Phi)^{-1} \Gamma + D}{1 - \Phi} U(z) \]

**Pole and Zeros**

- Poles and Zeros can be obtained from the denominator and numerator of the pulse-transfer function.
- A pole \( z = a \) corresponds to a free mode of the system associated with the sequence \( x[k] = a^k \).
- Poles are also the eigenvalues of the system matrix \( \Phi \).
- The zeros are related to how inputs and outputs are coupled to the states.
- Zeros can also be characterized by their signal blocking properties. A zero \( z = a \) means that a transmission \( u(k) = a^k \) is blocked by the system.
Consider the continuous-time system described by the $n$th-order state-space model
\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t)
\]
\[y(t) = Cx(t)\]

The poles of the system are the eigenvalues of $A$: $\lambda_i(A), i = 1, \ldots, n$. The zero-order-hold sampling gives the discrete-time system
\[
x[k+1] = \Phi x[k] + \Gamma u[k]
\]
\[y[k] = Cx[k]\]

The poles of this system are the eigenvalues of $\Phi$: $\lambda_i(\Phi), i = 1, \ldots, n$.

Because $\Phi = e^{A\Delta}$, it follows from the properties of matrix functions that $\lambda_i(\Phi) = e^{\lambda_i(A)\Delta}$.

This defines the mapping of the complex $s$-plane into the $z$-plane, when $z = e^{s\Delta}$.

**Complex Poles:**

Consider the continuous-time system
\[
\omega_0^2 s^2 + 2\zeta\omega_0 s + \omega_0^2.
\]

The poles of the corresponding discrete-time system are given by the characteristic equation
\[z^2 + a_1z + a_2 = 0\]

where
\[a_1 = -2\zeta e^{-\zeta\omega_0\Delta} \cos(\sqrt{1 - \zeta^2\omega_0^2}\Delta)\]
\[a_2 = e^{-2\zeta\omega_0\Delta}\]

This map is not bijective - several points in the $s$-plane are mapped into the same point in the $z$-plane (aliasing effect) ($\Delta \equiv h$).

Step responses of the second order discrete-time system for different values of $\Delta$ when $\zeta = 0.5$ and $\omega_0 = 1.83$, which gives a rise time $t_r = 1$ s. (a) $\Delta = 0.125\Delta$, (b) $\Delta = 0.25\Delta$, (c) $\Delta = 0.5\Delta$, and (d) $\Delta = 1\Delta$.

Loci of constant $\zeta$ (solid) and $\omega_0$ (dashed) when a second order continuous-time system is sampled.
**Zeros**

- It is not possible to give a simple formula for the mapping of zeros.
- The discrete-time system has in general \( n - 1 \) zeros. The sampling procedure gives extra zeros.
- For fast sampling periods, a discrete-time system will have zeros in
  \[
  z_i = e^{s_i \Delta}
  \]
  where the \( s_i \)'s are the zeros of the continuous-time system.
- A continuous-time system with a stable inverse may become a discrete-time system with an unstable inverse (poles outside the unit circle). Also, continuous-time nonminimum-phase system will not always become a discrete-time system with an unstable inverse.
- Large effect of sampling period on zeros.

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