Detecting and Enforcing Monotonicity for Hybrid Control Systems Synthesis

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Outline

1 Abstraction Based Hybrid Control Synthesis
2 Monotone Dynamical Systems
3 Monotonisation through feedback
4 Example
5 Summary
Abstraction Based Hybrid Control Synthesis

- Is applied to nonlinear continuous-time dynamical systems with quantised input/output signals.
- Continuous dynamics is “replaced” by discrete abstraction ⇝ the underlying hybrid control problem is converted into a purely discrete one.
- The modified problem can be efficiently solved using standard methods from discrete event systems (DES) theory.
Abstraction Based Hybrid Control Synthesis

Abstraction procedure *in theory*

- Partitioning of the state-space
- Mapping of the partition cell under the flow $\phi_u$
- Extraction of the discrete transition structure
Abstraction Based Hybrid Control Synthesis

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Abstraction procedure in practice

- Mesh generation
- Integration of the set of nodes under the flow $\phi_u$
- Approximation of the reachable set
- Approximation can be non-conservative!
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There are few classes of dynamical systems whose phase flow can be calculated exactly or can be safely overapproximated:

- Convexity preserving systems (e.g., linear systems)
- Monotone systems!
Partial Order Relation

Definition

Consider some set $P$ and a relation $\leq$ on $P$. Then $\leq$ is a partial order if it is reflexive, antisymmetric, and transitive.

- $a \leq a$ (reflexivity),
- $(a \leq b) \& (b \leq a) \Rightarrow a = b$ (antisymmetry),
- $(a \leq b) \& (b \leq c) \Rightarrow a \leq c$ (transitivity).

An order relation said to be total if for all distinct $a, b \in P$

- $a \leq b$ or $b \leq a$ (totality).

Set of real numbers $\mathbb{R}$

The set of reals is a totally ordered set.
Partial Order Relation

Set of real vectors $\mathbb{R}^n$

- The set of real vectors $x = \{x_1, \ldots, x_n\}' \in \mathbb{R}^n$ is a partially ordered set;
- The partial order relation can be introduced with respect to an arbitrary (convex pointed) cone,
- in particular, with respect to the orthant $\mathbb{R}_{\geq 0}^n$:
  - $x \preceq y$ (w.r.t. $\mathbb{R}^n_{\geq 0}$) if $x_i \leq y_i$, $i = 1, n$

Example of the partial ordering w.r.t. $\mathbb{R}^2_{\geq 0}$
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Partial Order Relation

Set of real vectors $\mathbb{R}^n$, cont.

- The partial ordering relation can also be defined w.r.t. an arbitrary orthant

$$\mathbb{R}_\delta^n = \{x \in \mathbb{R}^n | (-1)^{\delta_i} x_i \geq 0\},$$

where $\delta = \{\delta_1, \ldots, \delta_n\}, \delta_i \in \{0, 1\}$:

$x \preceq y \ (\text{w.r.t. } \mathbb{R}_\delta^n)$ if $(-1)^{\delta_i} x_i \leq (-1)^{\delta_i} y_i$

- Note, that $\mathbb{R}_\delta^n$ and $\mathbb{R}_\delta^n$ are equivalent up to change of sign.
Partial Order Relation

Set of real vectors $\mathbb{R}^n$, cont.

- The partial ordering relation can also be defined w.r.t. an arbitrary orthant

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- Note, that $\mathbb{R}^n_\delta$ and $\mathbb{R}^n_{\bar{\delta}}$ are equivalent up to change of sign.
Autonomous systems

Definition

Consider an autonomous (uncontrolled) dynamical system:

\[ \dot{x}(t) = f(x(t)). \] (*)

System (*) said to be monotone iff

\[ \forall x_1, x_2 \in \mathbb{R}^n, x_1 \preceq x_2 : \phi(t_0, x_1, t) \preceq \phi(t_0, x_2, t), t \geq t_0, \]

i.e., the ordered states remain ordered when time progresses.
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Assumed, the system is monotone w.r.t. $\mathbb{R}_\geq 0$

Abstraction procedure

- Rectangular partitioning
- 2 points are enough for $\mathbb{R}^n$!
- Overapproximation of the reachable set
- Conservativity is guaranteed!
### Autonomous systems

<table>
<thead>
<tr>
<th>Y₁</th>
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<th>Y₅</th>
</tr>
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<tbody>
<tr>
<td>Y₂</td>
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Autonomous systems

Monotonicity condition

- System (*) is monotone w.r.t. $\mathbb{R}_\delta$ on the convex set $D$ iff

$$(-1)^{\delta_i + \delta_j} \frac{\partial f_i}{\partial x_j}(x) \geq 0, \quad i \neq j, \quad x \in D.$$  

Monotonicity condition – examination

The above condition can be checked in two steps

1. Are the off-diagonal elements of the Jacobian matrix sign-stable and sign-symmetric?

\[
\left\{ \frac{\partial f_i(x)}{\partial x_j} \geq 0 \forall x \in D \right\} \lor \left\{ \frac{\partial f_i(x)}{\partial x_j} \leq 0 \forall x \in D \right\}
\]

\[
\frac{\partial f_i(x)}{\partial x_j} \cdot \frac{\partial f_j(x)}{\partial x_i} \geq 0 \forall x \in D
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If "yes", we proceed with the second step ...
Autonomous systems

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If “yes”, we proceed with the second step ...
Monotonicity condition - examination, \textit{cont.}

2. Check whether the Boolean equalities hold

\[ \delta_i \oplus \delta_j = s_{ij}, \quad i < j \]

where,

\[
s_{ij} = \begin{cases} 
0 & \text{if } \frac{\partial f_i(x)}{\partial x_j} > 0 \lor \left( \frac{\partial f_i(x)}{\partial x_j} = 0 \land \frac{\partial f_j(x)}{\partial x_i} > 0 \right) \\
1 & \text{if } \frac{\partial f_i(x)}{\partial x_j} < 0 \lor \left( \frac{\partial f_i(x)}{\partial x_j} = 0 \land \frac{\partial f_j(x)}{\partial x_i} < 0 \right) \\
\text{arbitrary in}\{0, 1\} & \text{if } \frac{\partial f_i(x)}{\partial x_j} = \frac{\partial f_j(x)}{\partial x_i} = 0.
\end{cases}
\]

characterise the signs of elements of the Jacobian matrix.
Autonomous systems

Example

Let us consider the Jacobian matrix with the following sign structure

$$J = \begin{bmatrix} * & + & 0 & - \\ + & * & + & 0 \\ 0 & + & * & 0 \\ - & 0 & 0 & * \end{bmatrix},$$

where,

0: $f'_{ij} = 0,$

+ : $f'_{ij} \geq 0,$

− : $f'_{ij} \leq 0,$

* : irrelevant.

⇒

$s_{12} = 0$  $s_{23} = 0$
$s_{13} \in \{0, 1\}$  $s_{24} \in \{0, 1\}$
$s_{14} = 1$  $s_{34} \in \{0, 1\}$

⇓

$\delta_1 \oplus \delta_2 = 0$
$\delta_1 \oplus \delta_3 \in \{0, 1\}$
$\delta_1 \oplus \delta_4 = 1$
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\begin{align*}
s_{12} &= 0 & s_{23} &= 0 \\
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### Autonomous systems

#### Example

Let us consider the Jacobian matrix with the following sign structure

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where,

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Autonomous systems

Graphical conditions for monotonicity


- Are based on the building of the incidence graph of the system.

- Are useful mainly for low-order systems, can hardly be used for analysis purposes.
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Autonomous systems

The system of Boolean equations

\[ \delta_i \oplus \delta_j = s_{ij}, \quad i < j \]  

is solvable w.r.t. \( \delta_i \) iff the following condition is satisfied:

\[ s_{ij} \oplus s_{ik} = s_{jk}, \quad i < j, \ j < k, \ i, j, k \leq n. \]  

Furthermore,

\[ \delta = \{0, s_{12}, \ldots, s_{1n}\} \]

is a solution of (*).
Autonomous systems

Algebraical condition for monotonicity – an alternative

- The system of Boolean equations

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Autonomous systems

Example, cont.

\[ s_{12} = 0 \]
\[ s_{13} \in \{0, 1\} \]
\[ s_{14} = 1 \]
\[ s_{23} = 0 \]
\[ s_{24} \in \{0, 1\} \]
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\[ s_{12} \oplus s_{23} = 0 \implies s_{13} = 0 \]
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Autonomous systems

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\[ \delta = \{0, 0, 0, 1\} \]
Autonomous systems

Example, \textit{cont.}

\begin{align*}
s_{12} &= 0 \\
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s_{23} &= 0 \\
s_{24} &\in \{0, 1\} \\
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\[ \delta = \{0,0,0,1\} \]
Controlled systems

Definition

The obtained results can be extended to dynamical systems driven by an exogenous input signal

\[ \dot{x}(t) = f(x(t), u(t)). \]  

A controlled dynamical system (*) is monotone w.r.t. the orthants \( \mathbb{R}^n_\delta \) and \( \mathbb{R}^m_\gamma \) if the following implication holds \( \forall t \geq 0 \):

\[ x_1 \preceq_\delta x_2, \ u_1(\tau) \preceq_\gamma u_2(\tau), \ 0 \leq \tau \leq t \Rightarrow \]

\[ \phi_t(x_1, u_{1\tau}) \preceq_\delta \phi_t(x_2, u_{2\tau}). \]
Controlled systems

Monotonicity condition

The controlled system (*) is monotone w.r.t. the orthants $\mathbb{R}_d^n$ and $\mathbb{R}_\gamma^m$ on the convex subset $D$ iff the following properties hold for all $x \in D$ and all $u \in U$:

\[
(-1)^{\delta_i+\delta_j} \frac{\partial f_i}{\partial x_j}(x,u) \geq 0, \quad i \neq j, \ i, j \leq n
\]

\[
(-1)^{\delta_i+\gamma_j} \frac{\partial f_i}{\partial u_j}(x,u) \geq 0, \quad i \leq n, \ j \leq m.
\]

Controlled systems

Algebraical condition for monotonicity

The controlled systems is monotone iff the following conditions are satisfied:

\[ s_{ij} \oplus s_{ik} = s_{jk}, \quad i < j, \ j < k, \ i, j, k \leq n, \]
\[ q_{ij} \oplus q_{kj} = s_{ik}, \quad i \neq k, \ i, k \leq n, \ j \leq m, \]

where, \( s_{ij} \) characterise the signs of the partial derivatives of \( f(x, u) \) w.r.t. \( x \) and \( q_{ij} \) the partial derivatives w.r.t. \( u \).
Moreover,

\[ \delta = \{0, s_{12}, \ldots, s_{1n}\}, \]
\[ \gamma = \{q_{11}, \ldots, q_{1m}\} \]

describe the respective orthants.
Obviously, monotone systems constitute a very small subset of the class of nonlinear systems. Thus, most practically relevant systems are not monotone :(  

\[\Downarrow\] 

Whether it is possible to use a feedback to enforce the monotonicity property for a given system?
Monotonisation through feedback

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Framework

In a hybrid control context, the control vector often consists of two components:

\[ u'(t) = [u'_1(t) \ u'_2(t)], \]

\[ u_1(t) = u_1(x(t)) \] – a continuous feedback signal,
\[ u_2(t) \] – a piecewise constant signal over the finite set \( U \).

The system can be treated separately on intervals, where \( u_2 \) is constant, i.e. \( u_2(t) = u_\kappa \in U, \ t \in [t_\kappa,t_{\kappa+1}) \).

- \( u_1 \) can be used to enforce the monotonicity.
- \( u_2 \) is considered as a parameter and does not affect the monotonicity property of the closed loop system.
Let’s consider the linear control system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t)
\end{align*}
\]

The Jacobian matrix is \( J = A \). Defining the control input as a linear function of the output, \( u(t) = KCx(t) \), we change the Jacobian to \( J = A + BK C \) and alter its sign structure accordingly.

Problem formulation

- How to find the appropriate orthant for the given system?
- How to choose the feedback matrix \( K \) to enforce the monotonicity property for the closed-loop system?
Linear case

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\[
\begin{bmatrix}
* & a_{12} & a_{13} & a_{14} \\
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\text{a}_{31} & \text{a}_{32} & * & a_{34} \\
\text{a}_{41} & \text{a}_{42} & \text{a}_{43} & * \\
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} \\
0 & 0 \\
b_{31} & b_{32} \\
b_{41} & b_{42} \\
\end{bmatrix}
\begin{bmatrix}
c_{11} & c_{12} & 0 & c_{14} \\
\end{bmatrix}
\]

\[
(-1)^{sq}(a_{qp} + \sum_{i=1}^{k} \sum_{j=1}^{l} b_{qi}^1 k_{ij} c_{jp}) \geq 0.
\]

- Analyse elements of matrices $B$ and $C$
- Find “uncontrollable” elements of the Jacobian
- Check these elements for sign-symmetry
- Deduce the required sign structure of the Jacobian and find elements with inappropriate signs
- Determine a feedback matrix $K$ to adjust these elements without changing the signs of the other entries.
## Linear case

The matrices $B$ and $C$ are defined as:

$$
\begin{bmatrix}
  * & a_{12} & a_{13} & a_{14} \\
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### Analyse elements of matrices $B$ and $C$

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### Check these elements for sign-symmetry

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### Determine a feedback matrix $K$ to adjust these elements without changing the signs of the other entries.

The inequality is:

$$(-1)^{s_{qp}}(a_{qp} + \sum_{i=1}^{k} \sum_{j=1}^{l} b_{i}^{1} k_{ij} c_{jp}) \geq 0.$$
Linear case

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  b_{31} & b_{32} \\
  b_{41} & b_{42} \\
\end{bmatrix}
\begin{bmatrix}
  c_{11} & c_{12} & 0 & c_{14} \\
\end{bmatrix}
\]

\[
(-1)^{s_{qp}}(a_{qp} + \sum_{i=1}^{k} \sum_{j=1}^{l} b_{qi}^{1} k_{ij} c_{jp}) \geq 0.
\]

- Analyse elements of matrices $B$ and $C$
- Find “uncontrollable” elements of the Jacobian
- Check these elements for sign-symmetry
- Deduce the required sign structure of the Jacobian and find elements with inappropriate signs
- Determine a feedback matrix $K$ to adjust these elements without changing the signs of the other entries.
Let’s consider a model of an activated sludge process.

\[
\begin{align*}
\frac{dX_b}{dt} &= \frac{Q_{in}}{V} X_{b,in} - \frac{Q_{out}}{V} X_b + \mu(S_s) X_b - bX_b \\
\frac{dS_s}{dt} &= \frac{Q_{in}}{V} S_{s,in} - \frac{Q_{out}}{V} S_s - \frac{1}{Y} \mu(S_s) X_b \\
\frac{dS_o}{dt} &= \frac{Q_{in}}{V} S_{o,in} - \frac{Q_{out}}{V} S_o - \frac{1-Y}{Y} \mu(S_s) X_b - bX_b
\end{align*}
\]

\[(*)\]

\(X_b, S_s \) and \(S_o\) are the concentrations of biomass, soluble substrate and dissolved oxygen. 
\(\mu(S_s)\) is the specific growth rate of the biomass described by Monod’s equation,

\[
\mu(S_s) = \frac{\bar{\mu} S_s}{K_s + S_s},
\]

\(Q_{in}\) and \(Q_{out}\) are the incoming and outgoing flows.
Using the conventional notation $x := [X_b, S_s, S_o]'$ and $u := [X_{b,in}, S_{s,in}, S_{o,in}, Q_{in}]'$ we rewrite the model equations

\begin{align*}
\dot{x}_1 &= \frac{u_1 u_4}{V} - \frac{Q_{out}}{V} x_1 + \mu(x_2)x_1 - bx_1 \\
\dot{x}_2 &= \frac{u_2 u_4}{V} - \frac{Q_{out}}{V} x_2 - \frac{1}{Y} \mu(x_2)x_1 \\
\dot{x}_3 &= \frac{u_3 u_4}{V} - \frac{Q_{out}}{V} x_3 - \frac{1-Y}{Y} \mu(x_2)x_1 - bx_1.
\end{align*}
The Jacobian matrix has the following form

$$\frac{Df}{DX} = \begin{bmatrix}
* & \frac{\bar{\mu}K_s x_1}{(K_s + x_2)^2} & 0 \\
-\frac{1}{Y} \frac{\bar{\mu}x_2}{K_s + x_2} & * & 0 \\
-\frac{1-Y}{Y} \frac{\bar{\mu}x_2}{K_s + x_2} - b & -\frac{1-Y}{Y} \frac{\bar{\mu}K_s x_1}{(K_s + x_2)^2} & *
\end{bmatrix}.$$  

We see that the partial derivatives $\frac{\partial f_1}{\partial x_2}$ and $\frac{\partial f_2}{\partial x_1}$ do not satisfy the sign-symmetry condition. The “right” sign is found from the analysis of the remaining elements of the Jacobian:

$s_{13} = 1, \ s_{23} = 1 \Rightarrow s_{12} = 0 \Rightarrow \frac{\partial f_2}{\partial x_1} \geq 0.$
The Jacobian matrix has the following form

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\end{bmatrix}.
\]

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The Jacobian matrix has the following form

\[
\frac{Df}{Dx} = \begin{bmatrix}
* & \frac{\overline{\mu}K_s x_1}{(K_s + x_2)^2} & 0 \\
-\frac{1}{Y} \frac{\overline{\mu} x_2}{K_s + x_2} & * & 0 \\
-\frac{1 - Y}{Y} \frac{\overline{\mu} x_2}{K_s + x_2} - b & -\frac{1 - Y}{Y} \frac{\overline{\mu} K_s x_1}{(K_s + x_2)^2} & *
\end{bmatrix}.
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We see that the partial derivatives \( \frac{\partial f_1}{\partial x_2} \) and \( \frac{\partial f_2}{\partial x_1} \) do not satisfy the sign-symmetry condition. The “right” sign is found from the analysis of the remaining elements of the Jacobian:

\[ s_{13} = 1, \; s_{23} = 1 \Rightarrow s_{12} = 0 \Rightarrow \frac{\partial f_2}{\partial x_1} \geq 0. \]
To adjust the Jacobian matrix we may use the control $u_2 = u_2(x(t))$. The Jacobian can be rewritten as:

$$
\frac{Df}{Dx} = \begin{bmatrix}
\frac{\partial u_2(x)}{V} & \frac{\partial u_2(x)}{\partial x_1} - \frac{1}{Y} \frac{\mu x_2}{K_s+x_2} & \frac{\bar{\mu}K_s x_1}{(K_s+x_2)^2} & 0 \\
-1 - Y \frac{\mu x_2}{Y} \frac{\bar{\mu} K_s x_1}{K_s+x_2} - b & -1 - Y \frac{\bar{\mu} K_s x_1}{Y (K_s+x_2)^2} & \frac{\partial u_2(x)}{\partial x_3}
\end{bmatrix}
$$

The control $u_2(x)$ has to be chosen to satisfy the following conditions $\forall x \in \mathbb{R}^3_{\geq 0}$, $u_4 \neq 0$:

$$
\frac{\partial u_2}{\partial x_1}(x) \geq \frac{V}{u_4 Y} \frac{\mu x_2}{K_s+x_2},
$$

$$
\frac{\partial u_2}{\partial x_3}(x) \leq 0.
$$
To adjust the Jacobian matrix we may use the control \( u_2 = u_2(x(t)) \). The Jacobian can be rewritten as:

\[
\frac{Df}{Dx} = \begin{bmatrix}
\frac{\bar{u}K_s x_1}{(K_s+x_2)^2} & 0 \\
\frac{u_4}{V} \frac{\partial u_2(x)}{\partial x_1} - \frac{1}{Y} \frac{\bar{\mu} x_2}{K_s+x_2} & * \\
-1 - Y \frac{\mu x_2}{Y} K_s + x_2 & b & \frac{1 - Y}{Y} \frac{\mu K_s x_1}{(K_s+x_2)^2} & *
\end{bmatrix}
\]

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\[
\frac{\partial u_2}{\partial x_3}(x) \leq 0.
\]
The above conditions define a family of control laws. In particular, a control law can be chosen as

\[ u_2(x) = c_1 x_1, \]

where \( c_1 = \frac{V\bar{\mu}}{u_4^* Y}, \ u_4^* = \min u_4. \)

- The system can be rendered monotone by a simple linear feedback.
- The monotonisation procedure does not require the measurement of all state variables.
The above conditions define a family of control laws. In particular, a control law can be chosen as

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- The system can be rendered monotone by a simple linear feedback.
- The monotonisation procedure does not require the measurement of all state variables.
Summary

- A simple and efficient algorithm to check whether an arbitrary continuous system is monotone with respect to some (a priori unknown) partial order relation has been provided.
- The proposed algorithm has been extended to the case of control systems.
- An approach to enforcing monotonicity with the help of feedback has been developed.

Further work

- Formulation of the Pontryagin Maximum Principle for monotone systems (with V. Azhmyakov)
- Study of special properties of monotone systems


Thank you!