Exercise 4

Exercise 4.1 (Dental clinic revisited)
Consider the Petri net model obtained for the dental clinic in part a) of Exercise 1.1, shown in Figure 1. Now, let the following limitations be imposed:

- A maximum of 3 people can be in the waiting room at the same time.
- There cannot be more than 4 patients inside the clinic (in the waiting room or being treated) at the same time.

Express these limitations as a specification for the system. Assuming that transitions $t_3$ and $t_5$ are observable but not controllable, and that $t_1$, $t_2$, and $t_4$ are controllable and observable, determine if the specification is ideally enforceable. Justify your answer based on Definition 2.11 from the Lecture Notes.

Figure 1: Petri net for the dental clinic.
Exercise 4.2

Figure 2: Petri net for Exercise 4.2.

The Petri net in Figure 2 is given.

a) Specify a controller that guarantees that the number of tokens in $p_5$ never exceeds 1 under the condition that all transitions are preventable (i.e., controllable) and observable.

b) Suppose, now, that $t_4$ is an uncontrollable transition. Show that, in this case, the specification in item a) is not ideally enforceable.

c) Obtain two other specifications, $(\Gamma_1, b_1)$ and $(\Gamma_2, b_2)$, that are both at least as strict as the original one and ideally enforceable. Use the algorithm given on the attached worksheet. What are the resulting controllers?
Exercise 4.3

The following Petri net (Figure 3) represents a robotic assembly cell in which piston rods are assembled into engine blocks. Table 1 contains a description of every place. The number of tokens in a place denotes the number of resources that are involved in an operation.

![Petri net for Exercise 4.3.](image)

Table 1: Description of the places of the Petri net from Figure 3.

<table>
<thead>
<tr>
<th>Place</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 )</td>
<td>Engine block and crankshaft are provided</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>S-380 robot aligns the crankshaft</td>
</tr>
<tr>
<td>( p_3 )</td>
<td>S-380 robot grabs and positions a new piston rod</td>
</tr>
<tr>
<td>( p_4 )</td>
<td>The engine block is getting prepared</td>
</tr>
<tr>
<td>( p_5 )</td>
<td>M1 robot picks up a tool</td>
</tr>
<tr>
<td>( p_6 )</td>
<td>M1 robot places the piston rod into the engine block and returns the tool</td>
</tr>
<tr>
<td>( p_7 )</td>
<td>M1 robot places a coverage on the piston rod</td>
</tr>
<tr>
<td>( p_8 )</td>
<td>M1 robot tightens the coverage</td>
</tr>
</tbody>
</table>

The following resources are available:

- 3 robots of type S-380
- 3 robots of type M1
- 2 tools

Express the boundedness of the resources in form of the inequality \( \Gamma x \leq b \). The transitions \( t_6, t_7, \) and \( t_8 \) are neither preventable nor observable. Is the desired specification ideally enforceable under these circumstances? If not, find another specification that is at least as strict as the previous one and ideally enforceable. Determine the incidence matrix of the controller (\( A_c \)) as well as the initial marking (\( x^0_c \)) of the controller places.\[1\]

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Algorithm — control of Petri nets with unpreventable and unobservable transitions

The hereby presented algorithm for determining the “stricter” specification, or the vector $r_1$ and the scalar $r_2$ in

\[
\begin{align*}
\gamma :&= r_1 + r_2 \gamma, \\
\beta :&= r_2(b + 1) - 1,
\end{align*}
\]

is taken from the book *Sistemi ad Eventi Discreti* by A. Di Febbraro and A. Giua (McGraw-Hill, 2002). The authors consider only simple specifications $\gamma^T x \leq b$, with $\gamma \in \mathbb{Z}^n$, $b \in \mathbb{Z}$, and only allow natural values for $r_1$ and $r_2$, that is, $r_1 \in \mathbb{N}^n$, $r_2 \in \mathbb{N}$. For the more general case of a specification $\Gamma x \leq b$ having the form shown in matrix inequality (2.15) from the lecture notes (page 29), it is always possible to apply the algorithm separately to one inequality $\gamma_i^T x \leq b_i$ at a time, with $i = 1, \ldots, q$.

The Case of Unpreventable Transitions

Given a Petri net $(N, x^0)$ with $n$ places and $m_{uc}$ unpreventable (uncontrollable) transitions, let $A_{uc}$ denote the part of the incidence matrix that corresponds to these transitions. The specification to be enforced is given by $\gamma^T x \leq b$. In order to determine $r_1$ and $r_2$ so that a stricter but ideally enforceable specification can be calculated according to (1), one constructs the following table of dimension $(n + 1) \times (m_{uc} + n + 1)$:

\[
C := \begin{bmatrix}
A_{uc} & I_{n \times n} & 0 \\
v^T & r_1^T & r_2
\end{bmatrix},
\]

where $I_{n \times n}$ is the identity matrix, $0$ is a null vector, and to start the algorithm the elements of the last row are defined as

\[
v^T := \gamma^T A_{uc}, \quad r_1 := 0, \quad r_2 := 1.
\]

If the specification $\gamma^T x \leq b$ is not ideally enforceable, this means $\gamma^T A_{uc} \not\leq 0$. Thus, add to the last row (or a positive multiple thereof) one of the previous rows (possibly multiplied by a positive integer) until vector $v$ has only elements $\leq 0$.

Formally, the algorithm for the case of unpreventable transitions consists of the following steps.

1. Construct the initial table according to (2).
2. Let $\mathcal{J} = \{ j \mid v(j) > 0 \}$ be the set of column indices corresponding to the positive entries of vector $v$. If $\mathcal{J} = \emptyset$, go to step 6.
3. If $\mathcal{J} \neq \emptyset$, select an index $\bar{j} \in \mathcal{J}$ and eliminate the element $v(\bar{j})$ as follows.
a) Determine the set of row indices that correspond to negative entries of column \( \bar{j} \) from \( A_{uc} \), \[ \mathcal{I} := \{ i \mid A_{uc}(i, \bar{j}) < 0 \} \]. If \( \mathcal{I} = \emptyset \), then it is not possible to eliminate the positive entry \( v(\bar{j}) \).
Go to step 5.

b) Select an index \( \bar{i} \in \mathcal{I} \) and compute \( d := \text{lcm}\{ |A_{uc}(\bar{i}, \bar{j})|, |v(\bar{j})| \} \), the least common multiple of the absolute values of the two elements \( A_{uc}(\bar{i}, \bar{j}) \) and \( v(\bar{j}) \).

c) Multiply the last row of the table by \( \frac{d}{|v(\bar{j})|} \) and add to it the \( \bar{i} \)-th row multiplied by \( \frac{d}{|A_{uc}(\bar{i}, \bar{j})|} \), i.e., set
\[
C(m + 1, \cdot) := \frac{d}{|v(\bar{j})|} C(m + 1, \cdot) + \frac{d}{|A_{uc}(\bar{i}, \bar{j})|} C(\bar{i}, \cdot).
\]
The \( \bar{j} \)-th element of vector \( v \) is now 0.

4. Go to step 2.

5. The algorithm ends with no result.

6. The algorithm ends and the last row of the table has the form
\[
\begin{bmatrix}
    v^T & r_1^T & r_2^T
\end{bmatrix}.
\]

In case the algorithm ends with step 6, then \( v \leq 0 \), whereas \( r_1 \in \mathbb{N}^n \), \( r_2 \in \mathbb{N} \). The stricter specification is thus given by (1), from which it can be shown that \( -\gamma^T A_{uc} = -v \geq 0 \) and therefore the new specification is ideally enforceable.

**The Case of Unpreventable and Unobservable Transitions**

A modified version of the presented algorithm also includes the case of unobservable transitions. The corresponding initial table in this case is
\[
C := \begin{bmatrix}
A_{uc} & A_{uouc} & I_{n \times n} & 0 \\
v^T & u^T & r_1^T & r_2^T
\end{bmatrix} = \begin{bmatrix}
A_{uc} & A_{uouc} & I_{n \times n} & 0 \\
\gamma^T A_{uc} & \gamma^T A_{uouc} & 0 \ldots 0 & 1
\end{bmatrix}
\] (3)
and one of the first \( n \) rows (or a positive multiple thereof) shall be added to the last row (possibly multiplied by a positive integer), until it holds that \( v \leq 0 \) and \( u = 0 \).

The steps for the case of unpreventable and unobservable transitions are the following.

1. Construct the initial table according to (3).

2. Let \( \mathcal{J} = \{ j \mid v(j) > 0 \} \). Apply the former version of the algorithm until \( \mathcal{J} = \emptyset \), meaning that \( v \leq 0 \).

3. Now, let \( \mathcal{L} = \{ \ell \mid u(\ell) \neq 0 \} \) be the set of column indices corresponding to the non-zero entries of vector \( u \). If \( \mathcal{L} = \emptyset \), go to step 7.

4. If \( \mathcal{L} \neq \emptyset \), select an index \( \bar{\ell} \in \mathcal{L} \) and eliminate the element \( u(\bar{\ell}) \) as follows.
   a) Determine the set of row indices that correspond to entries of column \( \bar{\ell} \) from \( A_{uouc} \) with opposite sign to \( u(\bar{\ell}) \), \( \mathcal{Z} := \{ z \mid \text{sgn}(A_{uouc}(z, \bar{\ell}) = -\text{sgn}(u(\bar{\ell})) \} \). If \( \mathcal{Z} = \emptyset \), then it is not possible to eliminate the non-zero entry \( u(\bar{\ell}) \). Go to step 6.

   b) Select an index \( \bar{z} \in \mathcal{Z} \) and compute \( d := \text{lcm}\{ |A_{uouc}(\bar{z}, \bar{\ell})|, |u(\bar{\ell})| \} \), the least common multiple of the absolute values of the two elements \( A_{uouc}(\bar{z}, \bar{\ell}) \) and \( u(\bar{\ell}) \).
c) Multiply the last row of the table by \( \frac{d}{u(\bar{\ell})} \) and add to it the \( \bar{z} \)-th row multiplied by \( \frac{d}{A_{\text{wouc}}(\bar{z}, \bar{\ell})} \), i.e., set

\[
C(m + 1, \cdot) := \frac{d}{|u(\bar{\ell})|} C(m + 1, \cdot) + \frac{d}{|A_{\text{wouc}}(\bar{z}, \bar{\ell})|} C(\bar{z}, \cdot).
\]

The \( \bar{\ell} \)-th element of vector \( u \) is now 0.

5. Go to step 2.

6. The algorithm ends with no result.

7. The algorithm ends and the last row of the table has the form

\[
\begin{bmatrix}
v^T & u^T & r_1^T & r_2^T
\end{bmatrix}.
\]

In case the algorithm ends with step 7, then \( v \leq 0 \) and \( u = 0 \), whereas \( r_1 \in \mathbb{N}^n \), \( r_2 \in \mathbb{N} \). The stricter specification is thus given by (1), from which it can be shown that \(-\gamma^T A_{\text{uc}} = -v \geq 0 \) and \(-\gamma^T A_{\text{wouc}} = -u = 0 \), and therefore the new specification is ideally enforceable.