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Output Reference Control for Weight-Balanced Timed Event Graphs

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Abstract— **Timed Event Graphs (TEGs) and their weighted extension WTEGs are particular timed Discrete Event Systems (DESs) where the dynamic behavior is described by synchronization and saturation effects. With dioids, a linear systems theory has been developed for (weighted) TEGs. In this paper, we use dioid theory to model the input-output behavior of a WTEG. Furthermore, we propose a control strategy which determines an optimal input for a predefined reference output. In this case, "optimal" means that input events are scheduled as late as possible with the restriction that the output events of the system do not occur later than specified by the reference. This strategy is often referred to as "just-in-time" control in literature.**

I. Introduction and Motivation

TEGs are a subclass of timed Petri nets in which each place has exactly one upstream and one downstream transition and all arcs have weight 1. The dynamic behavior of TEGs is governed by synchronization phenomena, which are clearly non-linear in traditional algebra. However, synchronization phenomena become linear in a particular algebra structure called dioids [1]. For this reason TEGs are popular models for systems where synchronization is essential such as manufacturing lines, computer systems and transportation networks. Over the last 30 years, this has led to the development of the (min,+)-linear system theory where basic concepts of traditional system theory such as state space representation, spectral analysis and transfer functions have been adapted to TEGs [1], [2], [3]. Moreover, along with the theory, formal methods for performance evaluation and controller synthesis have been introduced, among them model predictive control [4], state and output feedback control [5] and output reference control [1], [6], [7]. Weighted Timed Event Graphs (WTEGs) are extensions of TEGs where the weights on the arcs can take values in $\mathbb{N} = \{1, 2, \cdots\}$. Thus, WTEGs have more expressiveness and allow to describe a wider class of systems. The weights are suitable to express batch/split processes, for instance, when several occurrences of events are needed to induce a following event or when one event can result in several following events. Clearly, such batch and split processes are quite common in many manufacturing systems. Unlike TEGs, WTEGs have an event-variant behavior and can not be described by (min,+)-linear systems anymore [8]. However, some dedicated dioids have been introduced to deal with this event-variant behavior. For instance, in [8] a fluid version of WTEGs is investigated for which recurrent equations are introduced in a specific set of operators. Fluid WTEGs can be seen as a continuous approximation of the WTEGs discussed in this work. In [9] a slightly different dioid, denoted $\mathcal{E}^*[\![\delta]\!]$, is introduced for modeling and analysis of an important subclass of WTEGs - the class of WTEGs where parallel paths have balanced weights. This class is therefore called Weight-Balanced Timed Event Graphs (WBTEGs). It is shown that the input-output behavior of WBTEGs can be described by ultimately periodic series in $\mathcal{E}^*[\![\delta]\!]$. In [10] model matching control problems of WBTEGs in the dioid $\mathcal{E}^*[\![\delta]\!]$ are addressed.

The objective of this work is to introduce output reference control for the class of WBTEGs. For this control strategy, an output reference signal for a system is assumed to be *a priori* known, and the controller aims to schedule the input events of the system as late as possible, but under the restriction that output events do not occur later then specified by the reference signal. Thus, this approach is often referred to as "just-in-time" control. In the context of manufacturing systems this strategy leads to the minimization of the "work in process" and to the reduction of internal stocks. For ordinary TEGs, the first results on this control strategy have been published in [1], [2] and further extended in [6], [7] to consider the case of online updates of the reference signal. A first approach to solve the problem of output reference control for WTEG has been published in [11]. There, based on ideas from [8], the dynamics of WTEGs are modeled by recurrent equations in a state space representation and the optimal input is obtained by solving the "backward" equations for this state space representation.

In this work we address the problem of output reference control for a WBTEG based on its transfer function. By modeling the input signal for a WBTEG in the dioid $\mathcal{E}^*[\![\delta]\!]$ we show that the output of the WBTEG can be obtained by the product of its transfer function and the input signal. Likewise, we can describe a desired output signal for a system in the same dioid setting $\mathcal{E}^*[\![\delta]\!]$. Residuation theory [1] is used to solve the output reference control problem such that the input signal is optimized with respect to the "just-in-time" criterion.

The paper is organized as follows: Section II briefly introduces WTEGs and dioid theory. In Section III, the modeling process of WBTEGs in the dioid $\mathcal{E}^*[\![\delta]\!]$ is recalled. In Section IV, output reference control for WBTEGs is

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illustrated.

II. WTEG and Dioid

We assume that the reader is familiar with Petri net models (see, for instance [1], [12]). However, we briefly recall the basic concepts of WTEGs, which are a subclass of timed Petri nets. A WTEG is a directed bipartite graph defined by a 7-tuple $\mathcal{N} = \{P, T, A, w_u, w_d, \boldsymbol{M_0}, \boldsymbol{\phi}\}$, where

- $P = \{p_1, \cdots p_n\}$ is the finite set of places.
- $T = \{t_1, \cdots t_m\}$ is the finite set of transitions.
- $A \subseteq (T \times P) \cup (P \times T)$ is the set of arcs connecting transition to places and places to transitions. p_l is an upstream place of transition t_j (and t_j is a downstream transition of place p_l), if $(p_l, t_j) \in A$. Conversely, p_l is a downstream place of transition t_j (and t_j is an upstream transition of place p_l), if $(t_j, p_l) \in A$. For WTEGs each place p_l has exactly one upstream transition t_j and exactly one downstream transition t_o.
- $w_d : P \to \mathbb{N}$ and $w_u : P \to \mathbb{N}$ are the weights of the arcs (t_j, p_l), respectively (p_l, t_o).
- $\boldsymbol{M_0} \in \mathbb{N}_0^n$ is the vector of initial marking and $\boldsymbol{M_0}_l$ denotes the number of tokens initially in place p_l.
- $\boldsymbol{\phi} \in \mathbb{N}_0^n$ is the vector of holding times associated with places and $\boldsymbol{\phi}_l$ denotes the time a token must spend in place p_l before it becomes available to satisfy the firing condition of t_o.

The pair of arcs (t_j, p_l) and (p_l, t_o) constitutes a basic path, denoted by π_l, with a gain $\Gamma(\pi_l)$ defined by $\Gamma(\pi_l) := w_d(p_l)/w_u(p_l)$. A directed path is a sequence of basic paths, i.e, $\pi = \pi_{l_1} \cdots \pi_{l_q}$. The gain of a directed path π is then the product of all basic directed path gains, i.e., $\Gamma(\pi) = \prod_{j=1}^{q} \Gamma(\pi_{l_j})$.

An (ordinary) TEG is a WTEG where the weights of all arcs are 1, i.e., $\forall p_l \in P$, $w_d(p_l) = w_u(p_l) = 1$. A Weight-Balanced Timed Event Graph (WBTEG) is a WTEG with the additional restriction that parallel paths, i.e., paths beginning and ending in the same transition must have the same gain (they are weight balanced).

Fig. 1 shows a TEG, and Fig. 2 shows a WBTEG. In both figures, circles denote places, and numbers next to circles are the corresponding holding times. Bars denote transitions. Numbers next to arcs denote their weights. If no number is shown next to a place (resp. arcs), the corresponding holding time (resp. weight) is 0 (resp. 1). The *earliest functioning*

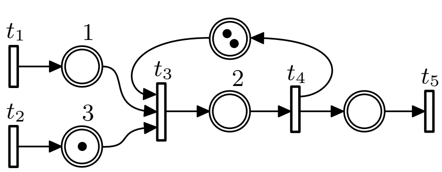


Fig. 1: A simple TEG.

of a WTEG is such that a transition t_i fires as soon as all upstream places p_l of t_i have at least $w_u(p_l)$ tokens and the corresponding holding times have passed. Then $w_u(p_l)$ tokens are removed from each upstream place p_l and $w_d(p_k)$ tokens are added to each downstream place p_k of t_i.

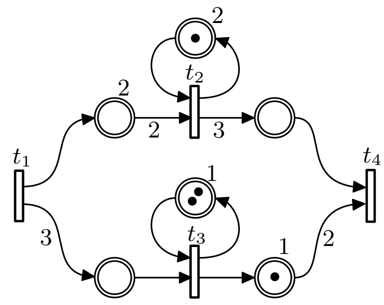


Fig. 2: A simple WBTEG.

We can divide the set of transitions of a WTEG into input, output and internal transitions. Input transitions are transitions without upstream places. Output transitions are transitions without downstream places, and all other transitions are called internal transitions.

A. Dioid Theory

In this section we briefly introduce dioids and recall the modeling process of a TEG in a particular dioid called (min,+)-algebra. For the modeling process of a TEG, a counter function $\mathcal{X} : \mathbb{Z} \to \mathbb{Z}_{min}\,(\mathbb{Z}_{min} := \mathbb{Z} \cup \{\pm\infty\})$ is associated to each transition. $\mathcal{X}(t)$ gives the accumulated number of firings of the transition before time t. A counter function is naturally a nondecreasing function.

Example 1: Consider the TEG of Fig. 1, by assigning the counter function $\mathcal{U}_1(t)$ (resp. $\mathcal{U}_2(t)$) to the input transition t_1 (resp. t_2), $\mathcal{X}_1(t)$ (resp. $\mathcal{X}_2(t)$) to internal transition t_3 (resp. t_4) and $\mathcal{Y}(t)$ to the output transition t_5, the *earliest functioning* of the TEG can be described by

$$\begin{aligned} &\mathcal{X}_1(t) = \min(\mathcal{X}_2(t) + 2, \mathcal{U}_1(t-1), \mathcal{U}_2(t-3) + 1), \\ &\mathcal{Y}(t) = \mathcal{X}_2(t) = \mathcal{X}_1(t-2). \end{aligned} \tag{1}$$

Formally, a dioid $\mathcal{D}$ is an algebraic structure with two binary operations, $\oplus$ (addition) and $\otimes$ (multiplication). Addition is commutative, associative and idempotent (i.e. $\forall a \in \mathcal{D},\ a \oplus a = a$). The neutral element for addition, denoted by ε, is absorbing for multiplication (i.e., $\forall a \in \mathcal{D},\ a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$). Multiplication is associative, distributive over addition and has a neutral element denoted by e. Note that, as in conventional algebra, the multiplication symbol $\otimes$ is often omitted. A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. In a complete dioid, the Kleene star of an element $a \in \mathcal{D}$ is defined by $a^* = \bigoplus_{i=0}^{\infty} a^i$ with $a^0 = \mathrm{e}$ and $a^{i+1} = a \otimes a^i$.

Theorem 1 ([1]): In a complete dioid $\mathcal{D}$, $x = a^*b$ is the least solution of the implicit equation $x = ax \oplus b$. ◁

A TEG can be conveniently modeled as a linear system in the particular dioid called the (min,+)-algebra. Formally, the latter is the set $\mathbb{Z}_{min}$ endowed with $\min$ as addition $\oplus$ and $+$ as multiplication $\otimes$, e.g., $5 \otimes 4 \oplus 7 = \min(5+4, 7) = 7$. Moreover, the zero element is $\varepsilon = \infty$, and the unit element is $\mathrm{e} = 0$, respectively. By convention $(\infty) \otimes (-\infty) = \infty$.

Example 2: In (min,+)-algebra, the system given in (1) is expressed as

$$\mathcal{X}_1(t) = 2\mathcal{X}_2(t) \oplus \mathcal{U}_1(t-1) \oplus 1\mathcal{U}_2(t-3),$$
$$\mathcal{Y}(t) = \mathcal{X}_2(t) = \mathcal{X}_1(t-2). \quad (2)$$

B. Dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

The dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, formally introduced in [1], [3], is useful to obtain transfer functions for TEGs. It is based on the time-shift operator δ^τ and the event-shift operator γ^ν, $(\tau,\nu \in \mathbb{Z})$ mapping counter functions to counter functions,

$$(\gamma^\nu \mathcal{X})(t) = \mathcal{X}(t) + \nu \text{ and } (\delta^\tau \mathcal{X})(t) = \mathcal{X}(t-\tau). \quad (3)$$

The operators γ^ν and δ^τ commute, i.e. $\gamma^\nu\delta^\tau = \delta^\tau\gamma^\nu$. Furthermore, addition is defined by:

$$((\gamma^\nu \oplus \gamma^{\nu'})\mathcal{X})(t) := (\gamma^\nu\mathcal{X})(t) \oplus (\gamma^{\nu'}\mathcal{X})(t),$$
$$((\delta^\tau \oplus \delta^{\tau'})\mathcal{X})(t) := (\delta^\tau\mathcal{X})(t) \oplus (\delta^{\tau'}\mathcal{X})(t).$$

This leads to the following simplification rules,

$$\gamma^\nu \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\nu')}, \quad \delta^\tau \oplus \delta^{\tau'} = \delta^{\max(\tau,\tau')}. \quad (4)$$

$\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is then the dioid of power series in γ and δ with Boolean coefficients $\{\tilde{\varepsilon},\tilde{\mathrm{e}}\}$ and exponents in $\mathbb{Z}$, with a quotient structure induced by the simplification rules (4). The unit element is denoted by $\mathrm{e} = \gamma^0\delta^0$ and the zero element is denoted by $\varepsilon = \bigoplus_{\nu,\tau\in\mathbb{Z}} \tilde{\varepsilon}\gamma^\nu\delta^\tau$.

Example 3: With the γ and δ operators, the system (2) can be expressed by $\mathcal{X}_1 = \gamma^2\mathcal{X}_2 \oplus \delta^1\mathcal{U}_1 \oplus \gamma^1\delta^3\mathcal{U}_2$, $\mathcal{Y} = \mathcal{X}_2 = \delta^2\mathcal{X}_1$, or equivalently, as a state space model, $\mathcal{X} = \boldsymbol{A}\mathcal{X} \oplus \boldsymbol{B}\mathcal{U}$; $\mathcal{Y} = \boldsymbol{C}\mathcal{X}$, where

$$\boldsymbol{A} = \begin{bmatrix} \varepsilon & \gamma^2 \\ \delta^2 & \varepsilon \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \delta^1 & \gamma^1\delta^3 \\ \varepsilon & \varepsilon \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \varepsilon & \mathrm{e} \end{bmatrix}.$$

Due to Theorem 1, the least solution for the output $\mathcal{Y}$ is given by $\mathcal{Y} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\mathcal{U} := \boldsymbol{H}\mathcal{U}$, which results in

$$\boldsymbol{H} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B} = \begin{bmatrix} \delta^3(\gamma^2\delta^2)^* & \gamma^1\delta^5(\gamma^2\delta^2)^* \end{bmatrix}.$$

$\boldsymbol{H}$ is called the transfer function matrix of the TEG which is composed of ultimately periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. In the case of a single-input and single-output (SISO) TEG the transfer relation is given by a series $h \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

There exists an isomorphism between counter functions $\mathcal{C} : \mathbb{Z} \to \mathbb{Z}_{min}$ and series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The counter functions $\mathcal{C}$ canonically associated with a series $c \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is such that $c = \bigoplus_{t\in\mathbb{Z}} \gamma^{\mathcal{C}(t)}\delta^t$ [13]. A detailed description of the transformation is given in [1](Chap. 5), and it is briefly recalled in [13]. Expressing counter functions as $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ series is convenient for calculations with transfer function models of TEGs in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. In the sequel, we denote a counter function by a capital calligraphic letter and the associated series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ by small letter, e.g., $\mathcal{C}(t)$ denotes the counter function canonically associated with the series $c \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. An impulse is a specific counter function $\mathcal{I}(t)$ defined as

$$\mathcal{I}(t) = \begin{cases} \mathrm{e} \quad (\text{resp. } 0) & \text{for } t \leq 0, \\ \varepsilon \quad (\text{resp. } +\infty) & \text{for } t > 0. \end{cases}$$

According to the representation of counter functions, this means that an infinity of events occur at time $t = 0$. The associated series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ to an impulse is given by the unit element $\mathrm{e} = \gamma^0\delta^0$. By expressing an input counter function $\mathcal{U} : \mathbb{Z} \to \mathbb{Z}_{min}$ as a series $u \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, the output counter $\mathcal{Y}(t)$, resp. y, of a SISO system can be obtained by the product of the transfer series $h \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and the input series u, i.e., $y = h \otimes u$, see [1], [13]. When we consider an impulse $\mathcal{I}(t)$ as the input of a system, we obtain $\mathcal{Y}(t) = h(\mathcal{I}(t))$, respectively $y = h \otimes \mathrm{e}$. Therefore, h is equivalent to the impulse response of the TEG. In general, let $c \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ be a series then $\mathcal{C}(t) = c(\mathcal{I}(t))$, i.e., the impulse response of the series c is equal to its associated counter function $\mathcal{C}(t)$. In [14] software tools are introduced for the computation of rational expressions of periodic series (matrices) in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

III. Modeling of WBTEGs in the Dioid $\mathcal{E}^*[\![\delta]\!]$

In contrast to TEGs, WBTEGs are not (min,+)-linear anymore. However, the dioid $\mathcal{E}^*[\![\delta]\!]$ introduced in [9] is suitable to model dynamic phenomena occurring in WBTEGs. In order to describe the event-variant behavior of WBTEGs, two new operators, event duplication μ_m and event division β_b, are introduced, besides the event-shift γ^ν and time-shift δ^τ operators, see (3). In analogy with the modeling process of TEGs in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, see subsection II-A, a counter function $\mathcal{X}_i : \mathbb{Z} \to \mathbb{Z}_{min}$, is associated to each transition t_i. The set of counter functions, denoted by Σ, has a semi-module structure where addition $\oplus$ and multiplication $\otimes$ are defined as follows:

$$\mathcal{X},\mathcal{Y} \in \Sigma, \ (\mathcal{X} \oplus \mathcal{Y})(t) := \min(\mathcal{X}(t),\mathcal{Y}(t)),$$
$$\lambda \in \mathbb{Z}_{min} \ (\lambda \otimes \mathcal{X})(t) := \lambda + \mathcal{X}(t).$$

The $\oplus$ operation induces an order relation on Σ, i.e., for $\mathcal{X},\mathcal{Y} \in \Sigma$, $\mathcal{X} \preceq \mathcal{Y} \Leftrightarrow \mathcal{X} \oplus \mathcal{Y} = \mathcal{Y}$. An operator is a map $\rho : \Sigma \to \Sigma$ which is linear if (a) $\forall \mathcal{X},\mathcal{Y} \in \Sigma : \rho(\mathcal{X} \oplus \mathcal{Y}) = \rho(\mathcal{X}) \oplus \rho(\mathcal{Y})$ and (b) $\lambda \otimes \rho(\mathcal{X}) = \rho(\lambda \otimes \mathcal{X})$. An operator is additive if (a) is satisfied. To simplify notation we sometimes write $\rho\mathcal{X}$ instead of $\rho(\mathcal{X})$.

Definition 1 (Basic E-(event) Operator): The following additive operators are called basic E-operators:

$$\nu \in \mathbb{Z}, \ \gamma^\nu : \forall \mathcal{X} \in \Sigma, \ (\gamma^\nu\mathcal{X})(t) = \mathcal{X}(t) + \nu, \quad (5)$$
$$b \in \mathbb{N}, \ \beta_b : \forall \mathcal{X} \in \Sigma, (\beta_b\mathcal{X})(t) = \lfloor \mathcal{X}(t)/b \rfloor, \quad (6)$$
$$m \in \mathbb{N}, \ \mu_m : \forall \mathcal{X} \in \Sigma, (\mu_m\mathcal{X})(t) = m \times \mathcal{X}(t), \quad (7)$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to $a \in \mathbb{Q}$. ◁

These operators satisfy the following relations [9],

$$\mu_m\gamma^n = \gamma^{n\times m}\mu_m, \quad \gamma^n\beta_b = \beta_b\gamma^{n\times b}. \quad (8)$$

Definition 2 (Dioid of E-operators $\mathcal{E}$): We denote by $\mathcal{E}$ the dioid of operators obtained by sums and compositions of operators in $\{\gamma^\nu, \beta_b, \mu_m\}$ with $\nu \in \mathbb{Z}$, and $b, m \in \mathbb{N}$, where for $\forall w_1, w_2 \in \mathcal{E}$, $\mathcal{X} \in \Sigma$, addition and multiplication are defined as $(w_1 \oplus w_2)(\mathcal{X}) = w_1(\mathcal{X}) \oplus w_2(\mathcal{X})$ and $(w_1 \otimes w_2)(\mathcal{X}) = w_1(w_2(\mathcal{X}))$. The identity operator is denoted by $\mathrm{e} : \forall \mathcal{X} \in \Sigma, \ (\mathrm{e}\mathcal{X})(t) = \mathcal{X}(t)$ and the zero element is denoted by $\varepsilon : \forall \mathcal{X} \in \Sigma, \ (\varepsilon\mathcal{X})(t) = \infty$. ◁

Note that $\mathcal{E}$ is a complete dioid [9]. Furthermore the identity operator can be expressed as: $\mathrm{e} = \gamma^0 = \mu_1 = \beta_1$.

Since E-operators only affect event numbering, an E-operator w can be described by a Counter-value to Counter-value (C/C) function $\mathcal{F}_w : \mathbb{Z}_{min} \to \mathbb{Z}_{min}, k_i \mapsto k_o$. It is an untimed representation where $k_i = \mathcal{X}(t)$ is an input counter value and k_o an output counter value, respectively. For instance, let $\mu_2\beta_3\gamma^1 \in \mathcal{E}$ then $(\mu_2\beta_3\gamma^1\mathcal{X})(t) = \lfloor(\mathcal{X}(t)+1)/3\rfloor 2$ which leads to $\mathcal{F}_{\mu_2\beta_3\gamma^1}(k_i) = \lfloor(k_i+1)/3\rfloor 2$, see Fig. 3a. Thus, there is an isomorphism between the set of E-operators and the set of (C/C) functions. The order relation over the dioid $\mathcal{E}$ corresponds to the order induced by the min operation on (C/C) functions. For $w_1, w_2 \in \mathcal{E}$, $\forall \mathcal{X} \in \Sigma, \forall t \in \mathbb{Z}$,

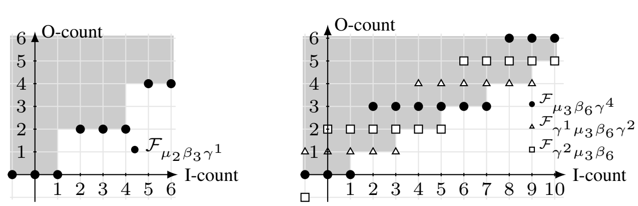
$$\begin{aligned} w_1 \succeq w_2 &\Leftrightarrow w_1 \oplus w_2 = w_1, \\ &\Leftrightarrow (w_1\mathcal{X})(t) \oplus (w_2\mathcal{X})(t) = (w_1\mathcal{X})(t), \\ &\Leftrightarrow \min((w_1\mathcal{X})(t), (w_2\mathcal{X})(t)) = (w_1\mathcal{X})(t), \\ &\Leftrightarrow \mathcal{F}_{w_1}(k) \leq \mathcal{F}_{w_2}(k),\ \forall k \in \mathbb{Z}_{min}. \end{aligned} \tag{9}$$

Definition 3 (Periodic E-operators): An E-operator $w \in \mathcal{E}$ is called (m,b)-periodic if $\forall k \in \mathbb{Z}_{min},\ \mathcal{F}_w(k+b) = m + \mathcal{F}_w(k)$, with $m, b \in \mathbb{N}$. The set of (m,b)-periodic E-operators is denoted by $\mathcal{E}_{m|b}$. ◁

For instance, $\gamma^2\mu_1\beta_2$ is a (1,2)-periodic E-operator in $\mathcal{E}_{1|2}$ and $\gamma^2\mu_1\beta_2 \notin \mathcal{E}_{1|3}$. Moreover, $\gamma^2\mu_1\beta_2 \oplus \mu_1\beta_1 \in \mathcal{E}$ is not a periodic operator. The (m,b)-periodic operator $\mu_m\beta_b$ can be extended to a multiple of its period in the following way

$$\mu_m\beta_b = \bigoplus_{i=0}^{n-1} \gamma^{im}\mu_{nm}\beta_{nb}\gamma^{(n-1-i)b}, \quad \text{where } n \geq 2. \tag{10}$$

For instance with $n = 3$, the operator $\mu_1\beta_2$ can be written as $\mu_3\beta_6\gamma^4 \oplus \gamma^1\mu_3\beta_6\gamma^2 \oplus \gamma^2\mu_3\beta_6$. Clearly $\mu_1\beta_2 \in \mathcal{E}_{1|2}$ and $\mu_1\beta_2 \in \mathcal{E}_{3|6}$ as well. Fig. 3b illustrates this extension of the $\mu_1\beta_2$ operator. The intersection of the areas beneath $\mathcal{F}_{\mu_3\beta_6\gamma^4}$, $\mathcal{F}_{\gamma^1\mu_3\beta_6\gamma^2}$ and $\mathcal{F}_{\gamma^2\mu_3\beta_6}$ is equal to the area beneath the (C/C) function $\mathcal{F}_{\mu_1\beta_2}$, i.e., $\mathcal{F}_{\mu_1\beta_2} = \min(\mathcal{F}_{\mu_3\beta_6\gamma^4}, \mathcal{F}_{\gamma^1\mu_3\beta_6\gamma^2}, \mathcal{F}_{\gamma^2\mu_3\beta_6})$.



(a) (C/C) func. $\mathcal{F}_{\mu_2\beta_3\gamma^1}$

(b) $\mathcal{F}_{\mu_1\beta_2}$ is equal to $\min(\mathcal{F}_{\mu_3\beta_6\gamma^4}, \mathcal{F}_{\gamma^1\mu_3\beta_6\gamma^2}, \mathcal{F}_{\gamma^2\mu_3\beta_6})$.

Fig. 3: (C/C) functions of periodic E-operators.

Proposition 1: Given two periodic operators $w_1 \in \mathcal{E}_{m_1|b_1}$, $w_2 \in \mathcal{E}_{m_2|b_2}$ such that $w_1 \neq \varepsilon$, $w_2 \neq \varepsilon$ and $\frac{m_1}{b_1} > \frac{m_2}{b_2}$. Then, w_1 and w_2 are not ordered, i.e., $w_1 \not\succeq w_2$ and $w_1 \not\preceq w_2$.

Proof: Due to (10) and by choosing $\bar{b} = lcm(b_1, b_2)$ we can represent $w_1 \in \mathcal{E}_{\bar{m}_1|\bar{b}}$ as an $(\bar{m}_1, \bar{b})$-periodic operator and $w_2 \in \mathcal{E}_{\bar{m}_2|\bar{b}}$ as an $(\bar{m}_2, \bar{b})$-periodic operator with corresponding periodic (C/C) functions

$$\mathcal{F}_{w_1}(k+\bar{b}) = \mathcal{F}_{w_1}(k) + \bar{m}_1, \quad \mathcal{F}_{w_2}(k+\bar{b}) = \mathcal{F}_{w_2}(k) + \bar{m}_2.$$

Then by evaluating the functions for $k = j\bar{b},\ j \in \mathbb{Z}$ we obtain

$$\mathcal{F}_{w_1}(j\bar{b}) = \mathcal{F}_{w_1}(0) + j\bar{m}_1, \quad \mathcal{F}_{w_2}(j\bar{b}) = \mathcal{F}_{w_2}(0) + j\bar{m}_2.$$

Since $\mathcal{F}_{w_1}(0)$ and $\mathcal{F}_{w_2}(0)$ are finite and $\bar{m}_1 > \bar{m}_2$ there exists a positive integer j such that $\mathcal{F}_{w_1}(j\bar{b}) > \mathcal{F}_{w_2}(j\bar{b})$ and a negative integer j such that $\mathcal{F}_{w_1}(j\bar{b}) < \mathcal{F}_{w_2}(j\bar{b})$. Thus the operators w_1 and w_2 are not ordered. ■

Proposition 2 ([9]): A periodic E-operator $w \in \mathcal{E}_{m|b}$ has a canonical form, which is a finite sum $w = \bigoplus_{i=1}^{I} \gamma^{n_i}\mu_m\beta_b\gamma^{n'_i}$ such that $0 \leq n'_i < b$, $n_i \in \mathbb{Z}$ and $I \leq b$.

A. Dioid $\mathcal{E}^*[\![\delta]\!]$

E-operators commute with the time-shift operator δ^t, i.e., $\forall w \in \mathcal{E},\ \delta^1 w = w\delta^1$. Thus we can define the dioid $\mathcal{E}^*[\![\delta]\!]$ as follows.

Definition 4 ([9]): (Dioid $\mathcal{E}^*[\![\delta]\!]$) We denote by $\mathcal{E}^*[\![\delta]\!]$ the quotient dioid in the set of formal power series in one variable δ with exponents in $\mathbb{Z}$ and coefficients in the non commutative complete dioid $\mathcal{E}$ induced by the equivalence relation $\forall s \in \mathcal{E}^*[\![\delta]\!],\ s = (\gamma^1)^* s = s(\gamma^1)^* = (\delta^{-1})^* s = s(\delta^{-1})^*$. ◁

The subset of $\mathcal{E}^*[\![\delta]\!]$ obtained by restricting the coefficients to $\mathcal{E}_{m|b}$, i.e. the set of (m,b)-periodic operators, is denoted by $\mathcal{E}^*_{m|b}[\![\delta]\!]$. For instance, $\mu_2\beta_3\gamma^1\delta^2 \in \mathcal{E}^*_{2|3}[\![\delta]\!]$, since $\mu_2\beta_3\gamma^1$ is $(2,3)$-periodic.

- A monomial in $\mathcal{E}^*_{m|b}[\![\delta]\!]$ is defined as $w\delta^\tau$ where $w \in \mathcal{E}_{m|b}$.
- A polynomial in $\mathcal{E}^*_{m|b}[\![\delta]\!]$ is a finite sum of monomials $p = \bigoplus_{i=1}^{I} w_i\delta^{t_i}$ such that $\forall i \in \{1, \cdots, I\},\ w_i \in \mathcal{E}_{m|b}$. For instance, the polynomial $\mu_2\beta_3\gamma^1\delta^2 \oplus \mu_3\beta_4\gamma^2\delta^3 \notin \mathcal{E}^*_{m|b}[\![\delta]\!]$.

Proposition 3 ([9]): Let $p = \bigoplus_{i=1}^{I} w_i\delta^{t_i} \in \mathcal{E}^*_{m|b}[\![\delta]\!]$, then p has a canonical form $p = \bigoplus_{j=1}^{J} w'_j\delta^{t'_j}$ such that w'_j are canonical and coefficients and exponents are strictly ordered, for $j \in \{1, \cdots, J-1\},\ t'_j < t'_{j+1}$ and $w'_j \succ w'_{j+1}$.

A series $s \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ is said to be ultimately periodic if it can be written as $s = p \oplus q(\gamma^\nu\delta^\tau)^*$, where $\nu, \tau \in \mathbb{N}_0$ and p, q are polynomials in $\mathcal{E}^*_{m|b}[\![\delta]\!]$, i.e, p and q have the same period. Note that a polynomial $p = \bigoplus_{i=1}^{I} w_i\delta^{t_i}$ can be considered as a specific ultimately periodic series where $\tau = 0$. An element $s \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ can be graphically represented in the 3D space (I-count/O-count/t-shift). For a series $s = \bigoplus_{i=1} w_i\delta^{t_i} \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ this graphical representation is constructed by depicting for every t_i the corresponding (C/C) function $\mathcal{F}_{w_i}$ of the coefficient w_i in the (I-count/O-count) plane of t_i. For instance, the graphical representation of $p = (\mu_3\beta_3\gamma^2 \oplus \gamma^1\mu_3\beta_3\gamma^1)\delta^2 \oplus \mu_3\beta_3\gamma^2\delta^3 \in \mathcal{E}^*_{3|3}[\![\delta]\!]$ is illustrated in Fig. 4, with the (I-count/O-count)-plane for $t = 2$ (resp. $t = 3$) shown in Fig. 5a (resp. Fig. 5b).

Proposition 4 ([9]):

- Let $s_1, s_2 \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ be two ultimately periodic series then $(s_1 \oplus s_2) \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ is an ultimately periodic series.

- Let $s_1 \in \mathcal{E}^*_{m_1|b_1}[\![\delta]\!]$ and $s_2 \in \mathcal{E}^*_{m_2|b_2}[\![\delta]\!]$ be two ultimately periodic series then $(s_1 \otimes s_2) \in \mathcal{E}^*_{m_1 m_2|b_1 b_2}[\![\delta]\!]$ is an ultimately periodic series.
- Let $s \in \mathcal{E}^*_{b|b}[\![\delta]\!]$ be an ultimately periodic series then $s^* \in \mathcal{E}^*_{b|b}[\![\delta]\!]$ is an ultimately periodic series.

Let us note that the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is a subdioid of $\mathcal{E}^*[\![\delta]\!]$, more precisely $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is the set $\mathcal{E}^*_{1|1}[\![\delta]\!]$, i.e., the set of $(1,1)$-periodic operators.

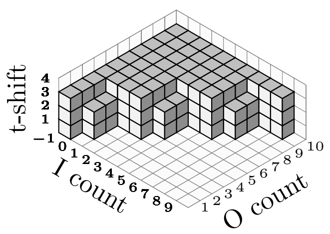
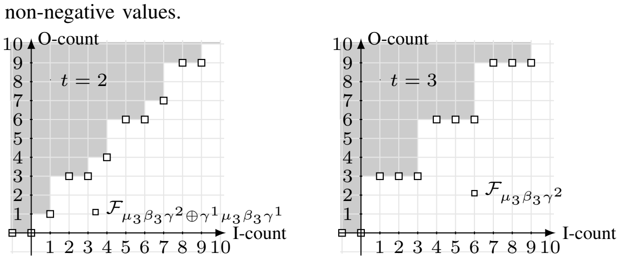


Fig. 4: 3D representation of polynomial $(\mu_3\beta_3\gamma^2 \oplus \gamma^1\mu_3\beta_3\gamma^1)\delta^2 \oplus \mu_3\beta_3\gamma^2\delta^3$. To improve readability, the 3D representation has been truncated to non-negative values.



(a) (I/O-count)-plane for $t = 2$ (b) (I/O-count)-plane for $t = 3$

Fig. 5: Slices of the coefficients in the (I/O-count)-plane of the polynomial $(\mu_3\beta_3\gamma^2 \oplus \gamma^1\mu_3\beta_3\gamma^1)\delta^2 \oplus \mu_3\beta_3\gamma^2\delta^3$

B. Transfer Function Matrices of WBTEGs

In [9] it is shown that E-operators associated with WBTEGs are periodic. Thus, for a WBTEG operating under the *earliest functioning* rule the firing relation between two transitions of a basic path $t_j \rightarrow p_l \rightarrow t_o$, can be described by operators in $\mathcal{E}^*[\![\delta]\!]$ as follows: $\mathcal{X}_o = \beta_{w_u(p_l)}\delta^{\boldsymbol{\phi}_l}\gamma^{\boldsymbol{M_{0}}_l}\mu_{w_d(p_l)}\mathcal{X}_j$, where $\mathcal{X}_j$ and $\mathcal{X}_o$ refers to the counter function of transition t_j and t_o, $w_d(p_l)$ and $w_u(p_l)$ are weights of the arc (t_j, p_l) and (p_l, t_o), $\boldsymbol{\phi}_l$ is the holding time of place p_l and $\boldsymbol{M_{0}}_l$ is the initial marking of p_l. As E-operators and the time-shift operator commute, $\beta_{w_u(p_l)}\delta^{\boldsymbol{\phi}_l}\gamma^{\boldsymbol{M_{0}}_l}\mu_{w_d(p_l)} = \beta_{w_u(p_l)}\gamma^{\boldsymbol{M_{0}}_l}\mu_{w_d(p_l)}\delta^{\boldsymbol{\phi}_l} \in \mathcal{E}^*_{m|b}[\![\delta]\!]$. For instance, take the WBTEG in Fig. 2, the firing relation between t_3 and t_4 corresponds to an operator representation $\mathcal{X}_4 = \beta_2\gamma^1\mu_1\delta^1\mathcal{X}_3$. Therefore, the firing relation between internal, input and output transition in a WBTEG can be described by a state space representation:

$$\mathcal{X} = \boldsymbol{A}\mathcal{X} \oplus \boldsymbol{B}\mathcal{U}, \qquad \mathcal{Y} = \boldsymbol{C}\mathcal{X},$$

where $\mathcal{X}$ (resp. $\mathcal{U}, \mathcal{Y}$) refers to counter functions of internal (resp. input, output) transitions and $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are matrices in $\mathcal{E}^*[\![\delta]\!]$ of appropriate size. In [9] it is shown that the transfer function matrix $\boldsymbol{H} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}$ of a WBTEG is composed of ultimately periodic series in $\mathcal{E}^*[\![\delta]\!]$.

Example 4: Consider the WBTEG in Fig. 2. The transfer function h describes the firing relation between input transition t_1 and output transition t_4 and is given by

$$\begin{aligned} h =& \mu_3\beta_2\delta^2 \oplus (\gamma^2\mu_3\beta_2\gamma^1 \oplus \gamma^3\mu_3\beta_2)\delta^3 \oplus \gamma^3\mu_3\beta_2\delta^4 \oplus \\ & (\gamma^4\mu_3\beta_2\gamma^1 \oplus \gamma^6\mu_3\beta_2)\delta^5 \oplus (\gamma^5\mu_3\beta_2\gamma^1 \oplus \gamma^6\mu_3\beta_2)\delta^6 \oplus \\ & (\gamma^1\delta^1)^*[(\gamma^6\mu_3\beta_2\gamma^1 \oplus \gamma^8\mu_3\beta_2)\delta^7]. \end{aligned}$$

A detailed calculation how to obtain the transfer function for this example is given in [9].

These transfer functions are useful to compute the output $\mathcal{Y}(t)$ for a given input counter $\mathcal{U}(t)$, i.e., $\mathcal{Y}(t) = h(\mathcal{U}(t))$. It is well known that the impulse response of an ordinary TEG describes its complete behavior [1]. However, this is not the case for WBTEGs, because they are event-variant systems. In [15] it is shown that the impulse response of a WBTEG with a transfer function $h = \bigoplus_i w_i\delta^{t_i} \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ can be obtained by

$$\bigoplus_i w_i\delta^{t_i}\mathcal{I}(t) = \bigoplus_i \gamma^{\mathcal{F}_{w_i}(0)}\delta^{t_i}\mathcal{I}(t).$$

This impulse response is a sum of time- and event-shifted impulses and gives us partial information about the transfer behavior of the WBTEG. Indeed it can be shown that the complete transfer behavior can be constructed from a finite set of event-shifted impulse responses, for a more exhaustive presentation see [15]. So far we can calculate the output of a WBTEG when the input is an impulse. To consider an arbitrary input counter $\mathcal{U}(t)$ we represent this counter as a sum of time- and event-shifted impulses. The output of the system is then obtained by the sum of these time- and event-shifted impulses responses. Differently stated, let $\mathcal{U}(t)$ be a counter with a corresponding series $u = \bigoplus_i \gamma^{\nu_i}\delta^{t_i} \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and $h \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ be the transfer function of a WBTEG, then

$$\mathcal{Y}(t) = h(\mathcal{U}(t)) = h\Big(\bigoplus_i \gamma^{\nu_i}\delta^{t_i}\mathcal{I}(t)\Big).$$

Since h is additive, $\mathcal{Y}(t) = \bigoplus_i h(\gamma^{\nu_i}\delta^{t_i}\mathcal{I}(t))$.

An alternative way to obtain the output of a WBTEG is to represent the input counter $\mathcal{U}(t)$ and the output counter $\mathcal{Y}(t)$ as series $u, y \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Since $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is a subdioid of $\mathcal{E}^*[\![\delta]\!]$, we can define the canonical injection from $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ into $\mathcal{E}^*[\![\delta]\!]$, denoted by

$$\text{Inj} : \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!] \rightarrow \mathcal{E}^*[\![\delta]\!],\ x \mapsto \text{Inj}(x) = x.$$

For instance, $\text{Inj}(\gamma^1\delta^2) = \gamma^1\mu_1\beta_1\delta^2 = \gamma^1\delta^2$. Furthermore, we define a mapping: $\Psi_{m|b} : \mathcal{E}^*_{m|b}[\![\delta]\!] \rightarrow \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

Definition 5: Let $s = \bigoplus_i w_i\delta^{t_i} \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ be an (m,b)-periodic series, then

$$\Psi_{m|b}(s) = \Psi_{m|b}\Big(\bigoplus_i w_i\delta^{t_i}\Big) = \bigoplus_i \gamma^{\mathcal{F}_{w_i}(0)}\delta^{t_i}.$$

◁

Remark 1: Let us note that for a transfer function $h \in \mathcal{E}^*_{m|b}[\![\delta]\!]$, $y_{\mathcal{I}} = \Psi_{m|b}(h)$ is the series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ associated with the impulse response $\mathcal{Y}_{\mathcal{I}}(t) = h(\mathcal{I}(t))$ of the WBTEG, i.e., $\mathcal{Y}_{\mathcal{I}}(t) = h(\mathcal{I}(t)) = \Psi_{m|b}(h)\mathcal{I}(t) = y_{\mathcal{I}}\mathcal{I}(t)$. ◁

Proposition 5: For a WBTEG with an (m,b)-periodic transfer function $h \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ and an input $u \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$, the output y is obtained by

$$y = \Psi_{m|b}(h \otimes \mathrm{Inj}(u)).$$

Proof:

$$\begin{aligned}\mathcal{Y}(t) &= h(\mathcal{U}(t)) = h(u(\mathcal{I}(t))) = h(\mathrm{Inj}(u)(\mathcal{I}(t)))\\ &= (h \otimes \mathrm{Inj}(u))(\mathcal{I}(t))\end{aligned}$$

Due to Remark 1, this is equivalent to $y = \Psi_{m|b}(h \otimes \mathrm{Inj}(u))$. ■

Example 5: Let us consider an input $u = \delta^1 \oplus \gamma^1\delta^4(\gamma^2\delta^2)^* \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ for a WBTEG with a transfer series $h = (\mu_3\beta_2\gamma^1 \oplus \gamma^2\mu_3\beta_2)\delta^1(\gamma^1\delta^1)^*$. For this input, the response $y \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ of the WBTEG is given by

$$\begin{aligned}y &= \Psi_{3|2}(h \otimes \mathrm{Inj}(u))\\ &= \Psi_{3|2}\left((\mu_3\beta_2\gamma^1 \oplus \gamma^2\mu_3\beta_2)\delta^1(\gamma^1\delta^1)^* \otimes \left(\delta^1 \oplus \gamma^1\delta^4(\gamma^2\delta^2)^*\right)\right)\\ &= \Psi_{3|2}\Big(\big((\mu_3\beta_2\gamma^1 \oplus \gamma^2\mu_3\beta_2)\delta^2 \oplus\\ &\qquad (\gamma^2\mu_3\beta_2\gamma^1 \oplus \gamma^3\mu_3\beta_2)\delta^5\big)(\gamma^1\delta^1)^*\Big)\\ &= (\delta^2 \oplus \gamma^2\delta^5)(\gamma^1\delta^1)^* = \delta^2 \oplus \gamma^1\delta^3 \oplus \gamma^2\delta^5(\gamma^1\delta^1)^*\end{aligned}$$

Thus, the output counter function $\mathcal{Y}(t) = y\mathcal{I}(t)$ is given by $\mathcal{Y}(t) = \min(\delta^2\mathcal{I}(t), \gamma^1\delta^3\mathcal{I}(t), \gamma^2\delta^5\mathcal{I}(t), \gamma^3\delta^6\mathcal{I}(t), \cdots)$ where, for $i \in \mathbb{N}_0$,

$$\delta^2\mathcal{I}(t) = \begin{cases}0 & \text{for } t \le 2,\\ +\infty & \text{for } t > 2,\end{cases} \quad \gamma^1\delta^3\mathcal{I}(t) = \begin{cases}1 & \text{for } t \le 3,\\ +\infty & \text{for } t > 3,\end{cases}$$

$$\gamma^{2+i}\delta^{5+i}\mathcal{I}(t) = \begin{cases}2+i & \text{for } t \le 5+i,\\ +\infty & \text{for } t > 5+i.\end{cases}$$

Therefore, $\mathcal{Y}(t \le 2) = 0,\ \mathcal{Y}(3) = 1,\ \mathcal{Y}(4) = 2,\ \mathcal{Y}(5+i) = 2+i$ with $i \in \mathbb{N}_0$. The first output event occurs at time $t = 2$ the next at $t = 3$, etc.

IV. Output Reference Control of WBTEG

Output reference control is an open loop control method. In a (WB)TEG context, the output reference signal - assumed to be *a priori* known - is defined by a counter function $\mathcal{Z}(t)$. The goal is to determine $\mathcal{U}(t)$, for a given system $h \in \mathcal{E}^*_{m|b}[\![\delta]\!]$, such that input events are scheduled as late as possible under the restriction that the output reference is met, i.e., $\mathcal{Z}(t) \succeq h(\mathcal{U}(t))$. Thus, at any instance of time there are at least as many event as specified in the counter function $\mathcal{Z}(t)$. Residuation theory offers an elegant way to solve this kind of problems for systems modeled in a dioid structure [1]. Indeed, residuation theory yields the greatest solution for the inequality $A \otimes X \preceq B$ and thus it is suitable to solve some best approximation control problems for TEGs [1]. In this section we show that the canonical injection Inj and the mapping $\Psi_{m|b}$ are residuated. Furthermore, we use these results to solve some best approximation control problems for WBTEGs.

A. Complete Dioids and Residuation Theory

Residuation theory is a formalism to address the problem of approximate mapping inversion over ordered sets [1]. The theory is suitable for complete dioids, since a complete dioid $\mathcal{D}$ is a partially ordered set, with a canonical order $\succeq$ defined by, $a, b \in \mathcal{D}$, $a \oplus b = a \Leftrightarrow a \succeq b$.

Definition 6 ([1](Chap. 4.4.2)): A mapping $f : \mathcal{D} \to \mathcal{L}$, with $\mathcal{D}$ and $\mathcal{L}$ complete dioids, is residuated if $\forall b \in \mathcal{L},\ f(x) \preceq b$ has a greatest solution in $\mathcal{D}$ denoted $f^\sharp(b)$. $f^\sharp$ is called the residual of f. ◁

Theorem 2 ([1](Chap. 4.4.2): A mapping $f : \mathcal{D} \to \mathcal{L}$, with $\mathcal{D}$ and $\mathcal{L}$ complete dioids, is residuated iff $f(\varepsilon) = \varepsilon$ and f is lower-semicontinuous, that is

$$f\left(\bigoplus_{x\in X} x\right) = \bigoplus_{x\in X} f(x),$$

for every (finite or infinite) subset X of $\mathcal{D}$. ◁

A residual mapping $f^\sharp$ satisfies the following inequality,

$$\forall y \in \mathcal{L}, \quad y \succeq f(f^\sharp(y)). \tag{11}$$

On a complete dioid the mapping $R_a : x \mapsto xa$, (right multiplication) resp. $L_a : x \mapsto ax$ (left multiplication) are lower-semicontinuous and therefore residuated. The residual mappings are denoted $R^\sharp_a(b) = b \phi a = \bigoplus\{x | xa \preceq b\}$ (right division by a) and $L^\sharp_a(b) = a \backslash\!\!\!\circ b = \bigoplus\{x | ax \preceq b\}$ (left division by a).

$\mathcal{E}^*[\![\delta]\!]$ is a complete dioid and one obtains the following results for the left (resp. right) division of periodic elements.

Proposition 6 ([9]):

- Let $s_1 \in \mathcal{E}^*_{m|b_1}[\![\delta]\!]$ and $s_2 \in \mathcal{E}^*_{m|b_2}[\![\delta]\!]$ be two ultimately periodic series then $(s_2 \backslash\!\!\!\circ s_1) \in \mathcal{E}^*_{b_2|b_1}[\![\delta]\!]$ is an ultimately periodic series.
- Let $s_1 \in \mathcal{E}^*_{m_1|b}[\![\delta]\!]$ and $s_2 \in \mathcal{E}^*_{m_2|b}[\![\delta]\!]$ be two ultimately periodic series then $(s_1 \phi s_2) \in \mathcal{E}^*_{m_1|m_2}[\![\delta]\!]$ is an ultimately periodic series.

Clearly the canonical injection Inj is lower-semicontinuous and thus Inj is residuated.

Proposition 7: Let $w\delta^\tau \in \mathcal{E}^*_{b|b}[\![\delta]\!]$ be a (b,b)-periodic monomial. Then $\mathrm{Inj}^\sharp(w\delta^\tau)$ is given by

$$\mathrm{Inj}^\sharp(w\delta^\tau) = \gamma^{\max_{k=0}^{b-1}(\mathcal{F}_w(k)-k)}\delta^\tau. \tag{12}$$

Proof: By definition of the residuated mapping $\mathrm{Inj}^\sharp(w\delta^\tau)$ is the greatest solution of the following inequality

$$w\delta^\tau \succeq \mathrm{Inj}(x) = \mathrm{Inj}\Big(\bigoplus_i \gamma^{\nu_i}\delta^{\zeta_i}\Big) = \bigoplus_i \gamma^{\nu_i}\delta^{\zeta_i}, \tag{13}$$

where $\bigoplus_i \gamma^{\nu_i}\delta^{\zeta_i} \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$. Clearly, the greatest ζ_i such that the inequality (13) holds is τ and thus,

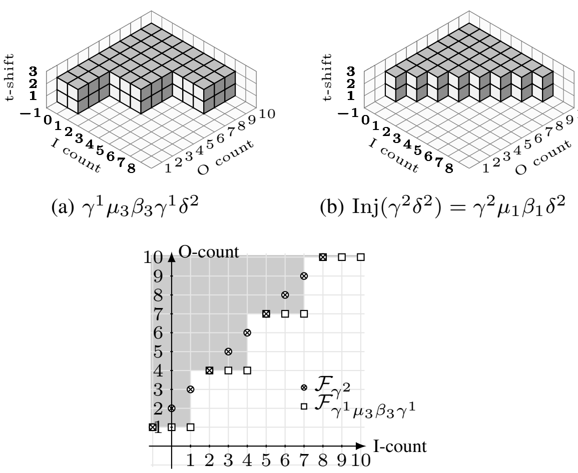
$$w\delta^\tau \succeq \bigoplus_i (\gamma^{\nu_i}\delta^\tau) = \gamma^\nu\delta^\tau, \quad \text{see, (4).} \tag{14}$$

Since $w\delta^\tau \succeq \gamma^\nu\delta^\tau \Leftrightarrow w \succeq \gamma^\nu$, it remains to find the smallest ν such that (14) holds. By considering the isomorphism between E-operators and (C/C) functions, see (9), this is equivalent to $\mathcal{F}_w(k) \le \mathcal{F}_{\gamma^\nu}(k),\ \forall k \in \mathbb{Z}_{min}$. Note that on the (C/C) functions the order is reversed. By using $\mathcal{F}_{\gamma^\nu}(k) = \nu + k$, see (5), we obtain

$$\mathcal{F}_w(k) \leq \nu + k \Leftrightarrow \nu \geq \mathcal{F}_w(k) - k, \quad \forall k \in \mathbb{Z}_{min}. \quad (15)$$

Since $\mathcal{F}_w$ is a (b,b)-periodic function it is sufficient to evaluate the function for $\forall k \in \{0, \cdots, b-1\}$. Therefore the smallest ν such that (15) (resp. (14)) holds is $\nu = \max_{k=0}^{b-1}(\mathcal{F}_w(k) - k)$. ■

Example 6: For the monomial $m = \gamma^1\mu_3\beta_3\gamma^1\delta^2 \in \mathcal{E}^*_{3|3}[\![\delta]\!]$, see Fig. 7a, the residual $\mathrm{Inj}^\sharp(m) = \gamma^{\max(1,0,2)}\delta^2 = \gamma^2\delta^2$, see Fig. 7b. Clearly $\mathrm{Inj}(\gamma^2\delta^2) \preceq \gamma^1\mu_3\beta_3\gamma^1\delta^2$ this is illustrated in Fig. 7c where the (C/C) functions for $t = 2$, i.e., $\mathcal{F}_{\gamma^1\mu_3\beta_3\gamma^1}$ and $\mathcal{F}_{\gamma^2}$, are shown. Obviously, $\mathcal{F}_{\gamma^2} \geq \mathcal{F}_{\gamma^1\mu_3\beta_3\gamma^1}$ ($\mathcal{F}_{\gamma^2} \preceq \mathcal{F}_{\gamma^1\mu_3\beta_3\gamma^1}$), in particular $\mathcal{F}_{\gamma^2}$ is the smallest (1,1)-periodic (C/C) function which is greater than $\mathcal{F}_{\gamma^1\mu_3\beta_3\gamma^1}$.



(a) $\gamma^1\mu_3\beta_3\gamma^1\delta^2$ (b) $\mathrm{Inj}(\gamma^2\delta^2) = \gamma^2\mu_1\beta_1\delta^2$

(c) (I/O-count)-plane

Fig. 7: Graphical illustration of $\mathrm{Inj}^\sharp(\gamma^1\mu_3\beta_3\gamma^1\delta^2) = \gamma^2\delta^2$.

Proposition 8: Let $s = \bigoplus_i w_i\delta^{\tau_i} \in \mathcal{E}^*_{b|b}[\![\delta]\!]$ be a (b,b)-periodic series in a canonical representation, see Prop. 3, extended to infinite sums, then

$$\mathrm{Inj}^\sharp(s) = \mathrm{Inj}^\sharp\left(\bigoplus_i w_i\delta^{\tau_i}\right) = \bigoplus_i \gamma^{\max_{k=0}^{b-1}(\mathcal{F}_{w_i}(k)-k)}\delta^{\tau_i}, \quad (16)$$

Second, for $s \in \mathcal{E}^*[\![\delta]\!]$ but $s \notin \mathcal{E}^*_{b|b}[\![\delta]\!]$,

$$\mathrm{Inj}^\sharp(s) = \varepsilon. \quad (17)$$

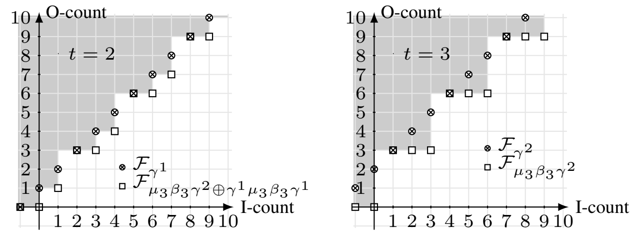
Proof: For (16): Consider $s = \bigoplus_i w_i\delta^{\tau_i}$ is in the canonical form, such that $\tau_i < \tau_{i+1}$ and $w_i \succ w_{i+1}$, then $\mathcal{F}_{w_i}$ is the (C/C) function in the plane for $t = \tau_i$ in the 3D representation of s. Thus, (14) must hold for every monomial $w_i\delta^{\tau_i}$ in the series $s = \bigoplus_i w_i\delta^{\tau_i}$.

To prove (17), recall that $\forall s \in \mathcal{E}^*[\![\delta]\!]$ we must satisfy the following inequality, see (11),

$$s \succeq \mathrm{Inj}\left(\mathrm{Inj}^\sharp(s)\right). \quad (18)$$

Now let us consider two series $s_1 \in \mathcal{E}^*_{m_1|b_1}[\![\delta]\!]$ and $s_2 \in \mathcal{E}^*_{m_2|b_2}[\![\delta]\!]$ such that $s_1 \neq \varepsilon$, $s_2 \neq \varepsilon$ and $\frac{m_1}{b_1} \neq \frac{m_2}{b_2}$. Then s_1 and s_2 are not ordered, i.e., $s_1 \not\succeq s_2$ and $s_1 \not\preceq s_2$ (see Prop. 1). The canonical injection $\mathrm{Inj}(\tilde{s})$ of an arbitrary series $\tilde{s} \in \mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$ is $(1,1)$-periodic, i.e., $\mathrm{Inj}(\tilde{s}) \in \mathcal{E}^*_{1|1}[\![\delta]\!]$. Thus, for $s \notin \mathcal{E}^*_{b|b}[\![\delta]\!]$, s and $\mathrm{Inj}(\tilde{s})$ are not ordered and (18) holds if and only if $\mathrm{Inj}^\sharp(s) = \varepsilon$. ■

Example 7: Consider the polynomial $p = \gamma^1\mu_3\beta_3\gamma^1\delta^2 \oplus \mu_3\beta_3\gamma^2\delta^3 \in \mathcal{E}^*_{3|3}[\![\delta]\!]$ with a canonical form $p = (\mu_3\beta_3\gamma^2 \oplus \gamma^1\mu_3\beta_3\gamma^1)\delta^2 \oplus \mu_3\beta_3\gamma^2\delta^3$. Then, $\mathrm{Inj}^\sharp(p) = \gamma^1\delta^2 \oplus \gamma^2\delta^3$. p is shown in Fig. 4. Fig. 8a illustrates $\mathrm{Inj}^\sharp((\mu_3\beta_3\gamma^2 \oplus \gamma^1\mu_3\beta_3\gamma^1)\delta^2) = \gamma^1\delta^2$ for the (I/O-count) plane at $t = 2$ and Fig. 8b illustrates $\mathrm{Inj}^\sharp(\mu_3\beta_3\gamma^2\delta^3) = \gamma^2\delta^3$ for the (I/O-count) plane at $t = 3$, respectively.



(a) (I/O-count)-plane for $t = 2$ (b) (I/O-count)-plane for $t = 3$

Fig. 8: Graphical illustration of $\mathrm{Inj}^\sharp(p) = \gamma^1\delta^2 \oplus \gamma^2\delta^3$.

The mapping $\Psi_{m|b}$ is by definition lower-semicontinuous, see Definition 5, therefore $\Psi_{m|b}$ is residuated.

Proposition 9: Let $s = \bigoplus_i \gamma^{\nu_i}\delta^{\tau_i} \in \mathcal{M}^{ax}_{in}[\![\gamma, \delta]\!]$. The residual $\Psi^\sharp_{m|b}(s) \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ of s is a series defined by

$$\Psi^\sharp_{m|b}\Big(\bigoplus_i \gamma^{\nu_i}\delta^{\tau_i}\Big) = \bigoplus_i \gamma^{\nu_i}\delta^{\tau_i}\mu_m\beta_b = s\mu_m\beta_b. \quad (19)$$

Proof: By definition of the residuated mapping, $\Psi^\sharp_{m|b}(\bigoplus_i \gamma^{\nu_i}\delta^{\tau_i}) \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ is the greatest solution of the following inequality

$$\bigoplus_i \gamma^{\nu_i}\delta^{\tau_i} \succeq \Psi_{m|b}(x) = \Psi_{m|b}\Big(\bigoplus_j w_j\delta^{\zeta_j}\Big), \quad (20)$$

where $x = \bigoplus_j w_j\delta^{\zeta_j} \in \mathcal{E}^*_{m|b}[\![\delta]\!]$. First we show that (19) satisfies (20) with equality.

$$\Psi_{m|b}\Big(\bigoplus_i \gamma^{\nu_i}\delta^{\tau_i}\mu_m\beta_b\Big) = \bigoplus_i \gamma^{\mathcal{F}_{\gamma^{\nu_i}\mu_m\beta_b}(0)}\delta^{\tau_i} = \bigoplus_i \gamma^{\nu_i}\delta^{\tau_i},$$

since $\mathcal{F}_{\gamma_i^\nu\mu_m\beta_b}(0) = \nu_i + \lfloor 0/b \rfloor m = \nu_i$, see (5),(7) and (6). Taking into account that $\Psi_{m|b}$ is isotone, it remains to show that $\bigoplus_i \gamma^{\nu_i}\delta^{\tau_i}\mu_m\beta_b$ is the greatest solution of

$$\bigoplus_i \gamma^{\nu_i}\delta^{\tau_i} = \Psi_{m|b}(x) = \Psi_{m|b}\Big(\bigoplus_j w_j\delta^{\zeta_j}\Big) = \bigoplus_j \gamma^{\mathcal{F}_{w_j}(0)}\delta^{\zeta_j}. \quad (21)$$

Clearly, to achieve equality we need $\zeta_j = \tau_i$ and $\mathcal{F}_{w_j}(0) = \nu_i$. Furthermore, we are looking for the greatest $w_j \in \mathcal{E}^*_{m|b}[\![\delta]\!]$, such that $\nu_i = \mathcal{F}_{w_j}(0)$. Due to the canonical form Prop. 2 we can write an (m,b)-periodic E-operator as $\bigoplus_{i=1}^b \gamma^{n_i}\mu_m\beta_b\gamma^{n'_i}$ with $0 \leq n'_i < b$. This operator corresponds to a (C/C) function

$$\mathcal{F}(k) = \min_{i=1}^{b}\Big(n_i + \Big\lfloor \frac{n'_i + k}{b} \Big\rfloor m\Big).$$

Now we examine $\mathcal{F}(k)$ for $k = 0$, thus

$$\mathcal{F}(0) = \min_{i=1}^{b}\Big(n_i + \Big\lfloor \frac{n'_i}{b} \Big\rfloor m\Big).$$

Recall that $0 \leq n'_i < b$, hence $\mathcal{F}_{w_j}(k) = \nu_i + \lfloor (0+k)/b \rfloor m$ is the smallest (m,b)-periodic (C/C) function such that (21) holds, i.e., $\mathcal{F}_{w_j}(0) = \mathcal{F}_{\gamma^{\nu_i}\mu_m\beta_b}(0) = \nu_i + \lfloor 0/b \rfloor m = \nu_i$. This function corresponds to the operator $\gamma^{\nu_i}\mu_m\beta_b$. ■

B. Output Reference Control

For a WBTEG with a transfer function $h \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ the output reference control problem can be stated by the inequality

$$\mathcal{Z}(t) \succeq h(\mathcal{U}(t)), \qquad (22)$$

where $\mathcal{Z}(t)$ is a counter function describing the desired schedule (*a priori* known signal) and $\mathcal{U}(t)$ is the unknown input that we want to optimize under the "just-in-time" criterion. Recall the calculation of a system output in Prop. 5 and the isomorphism between counter functions and $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ series, therefore we can rewrite (22) as

$$z \succeq \Psi_{m|b}(h \otimes \mathrm{Inj}(u)),$$

where z, u are series in $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ corresponding to the counter $\mathcal{Z}(t)$ and $\mathcal{U}(t)$. Note that for $h \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ and $u \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$, $\mathrm{Inj}(u) \in \mathcal{E}^*_{1|1}[\![\delta]\!]$ and thus the product $h \otimes \mathrm{Inj}(u) \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ (Prop. 4). In other words the periodicity of h and $h \otimes \mathrm{Inj}(u)$ are the same. The greatest solution for u is then given by

$$u_{opt} = \mathrm{Inj}^\sharp(h \lower.5ex\hbox{$\backslash$}\!\!\circ\, \Psi^\sharp_{m|b}(z)).$$

Since $h \in \mathcal{E}^*_{m|b}[\![\delta]\!]$ and $\Psi^\sharp_{m|b}(z) \in \mathcal{E}^*_{m|b}[\![\delta]\!]$, i.e., they have the same period, $\tilde{u} = h \lower.5ex\hbox{$\backslash$}\!\!\circ\, \Psi^\sharp_{m|b}(z) \in \mathcal{E}^*_{b|b}[\![\delta]\!]$ is (b,b)-periodic, see Prop. 6, which is the required form for a potential non zero solution of $\mathrm{Inj}^\sharp(\tilde{u})$, see Prop. 8.

Example 8: Consider a reference signal corresponding to the series $z = \delta^3 \oplus \gamma^3\delta^6(\gamma^1\delta^2)^* \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$, and the WBTEG with a transfer function $h \in \mathcal{E}^*_{3|2}[\![\delta]\!]$ given in Example 4. Then $\Psi^\sharp_{3|2}(z) = \mu_3\beta_2\delta^3 \oplus (\gamma^1\delta^2)^*[\gamma^3\mu_3\beta_2\delta^6]$ and $u_{opt} = \mathrm{Inj}^\sharp(h \lower.5ex\hbox{$\backslash$}\!\!\circ\, \Psi^\sharp_{3|2}(z)) = e \oplus \gamma^1\delta^1 \oplus \gamma^2\delta^4(\gamma^2\delta^6)^*$. The response y of the WBTEG to the optimal input u_{opt} is

$$y = \Psi_{3|2}(h \otimes u_{opt}) = \delta^3 \oplus [\gamma^3\delta^6 \oplus \gamma^5\delta^7](\gamma^3\delta^6)^*.$$

Fig. 9 illustrates the reference output $\mathcal{Z}(t)$ and the system output $\mathcal{Y}(t)$ resulting from the optimal input $\mathcal{U}_{opt}(t)$. Note that in (min,+), the order is reversed, one can see that, in Fig. 9 it is indeed true that $\mathcal{Z}(t) \succeq \mathcal{Y}(t)$.

V. Conclusion

In this work we solve the output reference control problem for WBTEGs. For this purpose, the transfer behavior of the WBTEG is modeled in the dioid $\mathcal{E}^*[\![\delta]\!]$. The proposed control method is based on residuation theory and provides the optimal control input under the "just-in-time" criterion. One possible extension of this work is to modify the control strategy such that online updates of the reference trajectory can be considered. This would allow the system to react to a change in customer demands, and will be considered in future work.

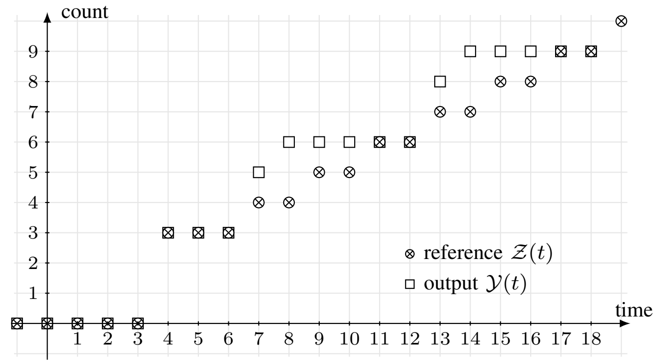


Fig. 9: Comparison between the reference output $\mathcal{Z}(t)$ and the system response $\mathcal{Y}(t)$ of the optimal input $\mathcal{U}_{opt}(t)$. As required, the condition $\mathcal{Z}(t) \succeq \mathcal{Y}(t)$ is satisfied.

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