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Brief paper

On the quadratic stability of switched linear systems associated with symmetric transfer function matrices☆

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ABSTRACT

In this paper we give necessary and sufficient conditions for weak and strong quadratic stability of a class of switched linear systems consisting of two subsystems, associated with symmetric transfer function matrices. These conditions can simply be tested by checking the eigenvalues of the product of two subsystem matrices. This result is an extension of the result by Shorten and Narendra for strong quadratic stability, and the result by Shorten et al. on weak quadratic stability for switched linear systems. Examples are given to illustrate the usefulness of our results.

1. Introduction

Consider the switched linear system

$$\Sigma_\sigma : \dot{x} = (1 - \sigma(t))A_1 x + \sigma(t) A_2 x, \quad \sigma(t) \in \{0, 1\}, \tag{1}$$

where A_1 and A_2 are constant matrices with real entries, and $\sigma(t)$ is an arbitrary time switching signal which assumes a finite number of switchings within a finite time interval. Let P be a symmetric positive definite matrix satisfying

$$A_i^T P + PA_i = -Q_i, \quad i \in \{1, 2\}. \tag{2}$$

Then, the function $V(x) = x^T Px$ is said to be a *strong* common quadratic Lyapunov function (CQLF) for the switched system (1) if the Q_i's are both positive definite. For the purpose of this paper, V is said to be a *weak* common quadratic Lyapunov function if both of the Q_i's are positive semi-definite and exactly one of them is singular. If such a CQLF exists the switched system (1) is called strongly quadratically stable implying that all solutions converge exponentially to zero, or weakly quadratically stable implying that all solutions are bounded (Note that strong quadratic stability is a sufficient condition for exponential stability of switched linear systems. For more results in this respect, see, *e.g.*, Hespanha and Morse (1999); Zhang and Huijun (2010).) In this paper, we wish to determine conditions on A_1 and A_2 such that a CQLF exists. Such stability problems arise in a variety of applications; see, for example, Liberzon (2003); Lin and Antsaklis (2009). Over the past decade many techniques have been developed to study CQLF existence problems. The most notable among these techniques are related to the use of Linear Matrix Inequalities (LMI) in the context of convex optimization. While LMI's and other numerical techniques are useful, usually they offer little insight into when such functions exist, and their extensions to cases where one or more of the system matrices is singular are problematic. Thus it is of interest to develop algebraic conditions that can be used to check for the existence of common quadratic Lyapunov functions. Initial results in this direction were given in Shorten, Corless, Wulff, Klinge, and Middleton (2009) and Shorten and Narendra (2003), where it was shown that any two matrices, one of which is Hurwitz and the other one has all eigenvalues in the open left half plane and exactly one eigenvalue at the origin, and differ by a rank-1 matrix, will admit a weak CQLF provided that the matrix product A_1A_2 has no real negative eigenvalues and exactly one zero eigenvalue. Despite much effort it has not been possible to develop similar results for more general matrix pairs. An alternative approach therefore is to seek pairs of matrices for which similar conditions hold true. In this paper we identify one such class.

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Specifically, we are interested in systems that arise in the context of symmetric transfer function matrices. Symmetric transfer function matrices are seen frequently, for example in the study of electrical circuits (Helmke, Rosenthal, & Wang, 2006; Semlyen & Gustavsen, 2009). Switched systems arise in situations where intermittent feedback is used to control such plants. The principal contribution in the note is to demonstrate that the Kalman–Yakubovic–Popov lemma is tight for such systems. We then use this observation to recover compact spectral conditions for the existence of a common (strong or weak) quadratic Lyapunov function for a certain class of switched systems. Finally, before proceeding it is important to note that even though this paper follows Shorten and Narendra (2003) and Shorten et al. (2009) in spirit (albeit for a much more general system class), the extension presented here does not immediately follow from these results, and is highly non-trivial involving detailed and original mathematical arguments.

Our paper is structured as follows. We conclude this section with the mathematical notation used in the paper. We then present our problem statement and some preliminary results that we shall use. We then present our main result and give some examples to illustrate their use.

Notation: Throughout, $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers, respectively. We denote n-dimensional real Euclidean space by $\mathbb{R}^n$ and the space of $n \times n$ matrices with real entries by $\mathbb{R}^{n\times n}$. We denote a state space representation of the $m \times m$ transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ by (A, B, C, D), where we always assume that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ has full column rank, $C \in \mathbb{R}^{m\times n}$ has full row rank, and $D \in \mathbb{R}^{m\times m}$ for some $n \geq m$. The inequality $Q \geq 0$ (respectively $Q > 0$) denotes that the matrix Q is positive semi-definite (respectively positive definite). $\otimes$ denotes the Kronecker product of two matrices, and v^* denotes the conjugate transpose of a vector $v \in \mathbb{C}^n$. Let y_{ij} be the (i, j) element of the matrix $Y \in \mathbb{R}^{n\times n}$. Then, we define the vectorization of Y as $\mathrm{vec}(Y) = [y_{11} \dots y_{n1}\ y_{12} \dots y_{n2} \dots y_{1n} \dots y_{nn}]^T$.

2. Problem statement: switched systems associated with symmetric transfer function matrices

We are interested in determining the existence of a CQLF for the following class of switched system: $\dot{x} = A(t)x$ where $A(t)$ is a matrix valued function taking the values A_1 or A_2 which are related through a symmetric transfer function matrix. Moreover, we assume that A_1 is Hurwitz (all eigenvalues have negative real part) and that A_2 has eigenvalues that are either in the open left half plane or at the origin. It is convenient to rewrite these matrices as $A_1 := A$, and $A_2 := A - BD^{-1}C$, and using this notation, to associate a transfer function matrix $G(s) = G^T(s) = C(sI - A)^{-1}B + D$ with the system. Thus, with this choice of A_1 and A_2 the switched system (1) is reformulated as

$$\Sigma_\sigma : \dot{x} = (A - \sigma(t)BD^{-1}C)x, \quad \sigma(t) \in \{0, 1\}. \tag{3}$$

Our interest in this paper is when $G(s)$ is symmetric for two principal reasons.

(i) Symmetric systems: First, symmetric transfer function matrices are ubiquitous in the study of electrical systems (Helmke et al., 2006; Semlyen & Gustavsen, 2009), and in systems with collocated sensors and actuators (Yang & Qiu, 2002). They are also found in the study of chemical process plants (Shinskey, 1984). The study of switched systems is important as symmetry of a transfer function matrix is often preserved under symmetric feedback. For example, consider the state space realization of a symmetric transfer function

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx. \end{cases} \tag{4}$$

Suppose now that intermittent output feedback of the form $u = -\sigma(t)Ky$ with $\sigma(t) \in \{0, 1\}$ and K a symmetric matrix is used to control the plant. Thus the closed loop system is of the form of $\dot{x} = (A - \sigma(t)BKC)x$ which is in the form of (3) if K is invertible. Such a scenario may readily occur whenever communication through which feedback is transmitted is unreliable. As is well known the stability of this system is not guaranteed, unless one can show the existence of a Lyapunov function.

(ii) Correspondence classes: A second motivation for the study of this system class comes from the definition of the matrices $A_1 = A$ and $A_2 = A - BD^{-1}C$. Many switched systems may be put in the form of this class by an appropriate choice of matrices B, C, and D. Thus, if we can establish results for the class of switched systems (3), then the same results can be used to determine quadratic stability of a much wider class of switched systems. Note, precisely which class of systems is isomorphic to the class considered in this paper is characterized by the following lemma (the proof is given in the Appendix).

Lemma 1. *Consider two matrices $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{n\times n}$. A sufficient condition for the existence of real matrices A, B, C, and $D = D^T > 0$ which satisfy $A_1 = A$, $A_2 = A - BD^{-1}C$, and $G(s) = G^T(s) = C(sI - A)^{-1}B + D$ is that the two matrices*

$$E_1 := I \otimes A_1 - A_1 \otimes I \quad \text{and}$$
$$E_2 := I \otimes A_2 - A_2 \otimes I \tag{5}$$

share a common eigenvector corresponding to a zero eigenvalue, say $vec(Y) = [y_{11} \dots y_{n1}\ y_{12} \dots y_{n2} \dots y_{1n} \dots y_{nn}]^T$, such that Y is symmetric and invertible, and $(A_1 - A_2)Y$ is positive semi-definite. Furthermore, if (A, B, C, D) is a minimal realization of $G(s)$, then this sufficient condition is also necessary.

3. Definitions and preliminary results

In this section we present several general results and definitions that are useful in proving the principal result of this note.

(i) *Strict positive realness:* An $m \times m$ rational transfer function matrix $G(s)$ is said to be strictly positive real (SPR) if there exists a real scalar $\alpha > 0$ such that $G(s)$ is analytic for $\mathrm{Re}(s) \geq -\alpha$ and

$$G(j\omega - \alpha) + G^T(-j\omega - \alpha) \geq 0, \quad \forall \omega \in \mathbb{R}; \tag{6}$$

see Corless and Shorten (2010) and Zhou, Doyle, and Glover (1996). The following characterization, inspired principally by Narendra and Taylor (1973), provides a more convenient description of a SPR transfer function matrix.

Lemma 2 (See Corless & Shorten, 2010)**.** *Suppose A is Hurwitz. Then the $m \times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is strictly positive real if and only if*

$$G(j\omega) + G^T(-j\omega) > 0, \quad \omega \in \mathbb{R}, \tag{7}$$

$$\lim_{\omega\to\infty} \omega^{2(m-p)} \det(G(j\omega) + G^T(-j\omega)) > 0, \tag{8}$$

where $p = rank(G(\infty) + G^T(\infty))$.

(ii) *Kalman–Yakubovic–Popov lemma (KYP):* A basic result in systems theory is the KYP lemma. The KYP lemma gives algebraic conditions for the existence of a certain type of Lyapunov functions; see, *e.g.*, Boyd, El Ghaoui, Feron, and Balakrishnan (1994).

Lemma 3 (KYP Lemma, See Boyd et al., 1994)**.** *Let A be Hurwitz, (A, B) be controllable, and (A, C) be observable. Then $G(s) = C(sI -$*

$A)^{-1}B+D$ is SPR if and only if there exist matrices $P=P^T>0$, L and W, and a number $\alpha>0$ satisfying

$$A^TP+PA=-L^TL-\alpha P \tag{9}$$

$$B^TP+W^TL=C \tag{10}$$

$$D+D^T=W^TW. \tag{11}$$

(iii) *Symmetric transfer function matrix:* We now define the class of systems which are of principal interest in this note; namely the class of matrices defined by the following lemma.

Lemma 4 (*See, e.g., Kouhi, Bajcinca, Raisch, & Shorten, 2013*). *Let (A,B,C,D) be a state space realization of a transfer function matrix $G(s)=C(sI-A)^{-1}B+D$. Then, $G(s)$ is symmetric, i.e. $G(s)=G^T(s)$, if and only if $D=D^T$ and*

$$CA^iB=(CA^iB)^T,\quad i=0,1,\ldots,n-1. \tag{12}$$

Proof. By referring to (Kailath, 1980, pp. 67), and by defining the characteristic polynomial of A as

$$\det(\lambda I-A)=\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_n,$$

the following identity holds

$$C(sI-A)^{-1}B=\frac{1}{\det(sI-A)}[s^{n-1}(CB)$$
$$+s^{n-2}(CAB+a_1CB)+\cdots+(CA^{n-1}B+\cdots+a_{n-1}CB)].$$

It is immediately evident that condition (12) and the assumption $D=D^T$ are necessary and sufficient for transfer function matrix $G(s)=C(sI-A)^{-1}B+D$ to be symmetric.

Symmetric transfer function matrices and transfer functions share many properties. In particular, we note the following:

Lemma 5. *For any symmetric transfer function matrix $G(s)=C(sI-A)^{-1}B+D$, the following equality holds:*

$$\frac{1}{2}\{G(j\omega)+G^T(-j\omega)\}=D-C(\omega^2I+A^2)^{-1}AB. \tag{13}$$

Proof. Utilizing the following identity for A Hurwitz

$$\frac{1}{2}\left\{(j\omega I-A)^{-1}+(-j\omega I-A)^{-1}\right\}=-(\omega^2I+A^2)^{-1}A,$$

we have

$$\frac{1}{2}\{G(j\omega)+G^T(-j\omega)\}=\frac{1}{2}\{G(j\omega)+G(-j\omega)\}$$
$$=\frac{1}{2}\{D+C(j\omega I-A)^{-1}B\}+\frac{1}{2}\{D+C(-j\omega I-A)^{-1}B\}$$
$$=D+\frac{1}{2}C\{(j\omega I-A)^{-1}+(-j\omega I-A)^{-1}\}B$$
$$=D-C(\omega^2I+A^2)^{-1}AB.$$

Finally, we conclude this section by giving some preliminary results concerning the strict positive realness of symmetric transfer function matrices.

Symmetric systems with nonsingular D: For symmetric transfer function matrices with positive definite D we have the following simple results.

Lemma 6 (*see Kouhi et al. (2013); Semlyen and Gustavsen (2009)*). *Given a Hurwitz matrix A, the symmetric transfer function matrix $G(s)=C(sI-A)^{-1}B+D$ with $D=D^T>0$ is SPR if and only if $A(A-BD^{-1}C)$ has no real negative eigenvalue.*

Proof of sufficiency: Suppose that $A(A-BD^{-1}C)$ has no real negative eigenvalue. By continuity of $\det(\omega^2I+A(A-BD^{-1}C))$ with respect to ω everywhere, we can write

$$\det(\omega^2I+A(A-BD^{-1}C))>0$$
$$\Rightarrow \det(\omega^2I+A^2)\det(I-(\omega^2I+A^2)^{-1}ABD^{-1}C)>0$$
$$\Rightarrow \det(\omega^2I+A^2)\det(I-C(\omega^2I+A^2)^{-1}ABD^{-1})>0$$
$$\Rightarrow \det(\omega^2I+A^2)\det(D^{-1})\det(D-C(\omega^2I+A^2)^{-1}AB)>0. \tag{14}$$

Note that in the third line of the above argument we used the general identity $\det(I-XY)=\det(I-YX)$. As A has no eigenvalue on the $j\omega$ axis, the identity $\det(\omega^2I+A^2)=\det(j\omega I+A)\det(-j\omega I+A)$ implies that $\det(\omega^2I+A^2)\neq 0$, for all $\omega\in\mathbb{R}$. Then, by continuity of $\det(\omega^2I+A^2)$ with respect to ω everywhere, we can deduce $\det(\omega^2I+A^2)>0$, for all $\omega\in\mathbb{R}$. On the other hand, the assumption that D is positive definite implies that D^{-1} is also positive definite and that $\det(D^{-1})>0$. Consequently, by (14) we have that $\det(D-C(\omega^2I+A^2)^{-1}AB)>0$ for all $\omega\in\mathbb{R}$, and then using the fact that $G(j\omega)$ is symmetric we can conclude from Lemma 5 that

$$\det\left(\frac{1}{2}\{G(j\omega)+G^T(-j\omega)\}\right)>0. \tag{15}$$

Furthermore, $G(j\omega)+G^T(-j\omega)$ is a Hermitian matrix implying that its eigenvalues are all real. Therefore, if for some frequency it fails to be positive definite, then there must exist an $\omega=\omega_1$ such that at least one eigenvalue of this matrix equals zero, that is $\det\left(\frac{1}{2}\{G(j\omega_1)+G^T(-j\omega_1)\}\right)=0$. This is obviously in contradiction with (15). Hence, using Lemma 2 with $p=m$, strict positive realness of $G(s)$ is established.

Proof of necessity: Suppose $G(s)$ is SPR. Then from Lemmas 2 and 5

$$\det\left(\frac{1}{2}\{G(j\omega)+G^T(-j\omega)\}\right)>0$$
$$\Rightarrow \det\left(D-C(\omega^2I+A^2)^{-1}AB\right)>0,$$

for all $\omega\in\mathbb{R}$. Using the fact that $\det(\omega^2I+A^2)>0$ for $\omega\in\mathbb{R}$, $\det(D^{-1})>0$, and following the computation (14) in reverse, we have $\det(\omega^2I+A(A-BD^{-1}C))>0$. This verifies that $A(A-BD^{-1}C)$ has no real negative eigenvalue.

Symmetric systems with singular D: Now, consider the symmetric transfer function matrix $G(s)=C(sI-A)^{-1}B+D$ with a singular matrix $D=D^T\geq 0$. Our aim is to study under which conditions $G(s)$ is SPR. Obviously, we cannot employ Lemma 6 as D may not be invertible. Therefore, we look for an alternative eigenvalue condition by investigating the transfer function matrix $G(\frac{1}{s})$, similar to the idea proposed in Shorten et al. (2009).

Lemma 7. *Suppose A is Hurwitz. Then, a symmetric transfer function matrix $G(s)=C(sI-A)^{-1}B+D$, with $D=D^T\geq 0$ and $\mathrm{rank}(D)=p\leq m$, is SPR if and only if $D-CA^{-1}B>0$, and $M:=A^{-1}\left(A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1}\right)$ has no real negative eigenvalue and has exactly $m-p$ eigenvalues equal to zero.*

Proof. Lemma 7 follows by developing the ideas in Shorten et al. (2009). A proof is given here for completeness. An alternative proof can also be found in Bajcinca and Voigt (2013).

Proof of necessity: Suppose $G(s)$ is SPR. Then, inequality (7) in Lemma 2 must hold. In particular for $\omega=0$

$$\frac{1}{2}\{G(0)+G^T(0)\}=G(0)=D-CA^{-1}B>0. \tag{16}$$

For the rest of the proof, it is convenient to study the transfer function matrix $G(\frac{1}{s})$ instead of $G(s)$. If (A,B,C,D) is a realization of $G(s)$ and A is Hurwitz, then a realization of $G(\frac{1}{s})$ is

$$(\bar{A},\bar{B},\bar{C},\bar{D})=(A^{-1},-A^{-1}B,CA^{-1},D-CA^{-1}B); \tag{17}$$

see Shorten et al. (2009) and the references therein. Note that $\bar{A} = A^{-1}$ is Hurwitz if A is Hurwitz. Referring again to Lemma 2 and substituting ω by $-\frac{1}{\omega}$, the following conditions must be valid

$$G\left(\tfrac{1}{j\omega}\right) + G^T\left(-\tfrac{1}{j\omega}\right) > 0, \quad \forall\omega \in \mathbb{R} \setminus \{0\}, \tag{18}$$

$$\lim_{\omega\to 0} \frac{1}{\omega^{2(m-p)}} \det\left(G\left(\tfrac{1}{j\omega}\right) + G^T\left(-\tfrac{1}{j\omega}\right)\right) > 0, \tag{19}$$

where $p = \operatorname{rank}(G(\infty) + G^T(\infty)) = \operatorname{rank}(D)$. Now, for the symmetric transfer function matrix $G(\frac{1}{s})$, we can exploit Lemma 5 by writing the following identity

$$\frac{1}{2}\left\{G\left(\tfrac{1}{j\omega}\right) + G^T\left(-\tfrac{1}{j\omega}\right)\right\} = \bar{D} - \bar{C}(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{B}. \tag{20}$$

Then, (18), (19), and (20) imply that

$$\det(\bar{D} - \bar{C}(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{B}) > 0, \quad \forall\omega \in \mathbb{R} \setminus \{0\}, \tag{21}$$

$$\lim_{\omega\to 0} \frac{1}{\omega^{2(m-p)}} \det(\bar{D} - \bar{C}(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{B}) > 0. \tag{22}$$

Furthermore, as $\bar{D} = D - CA^{-1}B > 0$ (see Eq. (16)), we can write

$$\begin{aligned}
&\det(\bar{D} - \bar{C}(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{B}) \\
&\quad = \det(\bar{D})\det(I - \bar{C}(\omega^2 I + \bar{A}^2)^{-1}\bar{A}\bar{B}\bar{D}^{-1}) \\
&\quad = \det(\bar{D})\frac{\det\left(\omega^2 I + \bar{A}(\bar{A} - \bar{B}\bar{D}^{-1}\bar{C})\right)}{\det\left(\omega^2 I + \bar{A}^2\right)} \\
&\quad = \det(\bar{D})\frac{\det(\omega^2 I + M)}{\det\left(\omega^2 I + \bar{A}^2\right)} > 0.
\end{aligned} \tag{23}$$

As $\bar{A} = A^{-1}$ is Hurwitz it does not have any eigenvalue on the $j\omega$ axis. Therefore, with continuity of $\det(\omega^2 I + \bar{A}^2)$ everywhere in $\omega \in \mathbb{R} \setminus \{0\}$, we have $\det(\omega^2 I + \bar{A}^2) > 0$. Consequently, (21) and (22) are equivalent to

$$\det\left(\omega^2 I + M\right) > 0, \quad \forall\omega \in \mathbb{R} \setminus \{0\}, \tag{24}$$

$$\lim_{\omega\to 0} \frac{1}{\omega^{2(m-p)}} \det(\omega^2 I + M) > 0. \tag{25}$$

Now, setting $\lambda = -\omega^2$, (24) and (25) lead to

$$\det(\lambda I - M) \neq 0, \quad \text{for } \lambda \in \mathbb{R}^- \setminus \{0\},$$

$$\lim_{\lambda\to 0} \frac{1}{\lambda^{m-p}} \det(\lambda I - M) \neq 0,$$

which means M does not have any real negative eigenvalue and at most has $m - p$ zero eigenvalues. Furthermore, as $\operatorname{rank}(D) = p$, there exists a matrix $D^\perp \in \mathbb{R}^{m\times(m-p)}$ such that $DD^\perp = 0$, and consequently $-CA^{-1}BD^\perp = (D - CA^{-1}B)D^\perp$. Then, we have

$$\begin{aligned}
MBD^\perp &= A^{-1}(A^{-1}B + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}B)D^\perp \\
&= A^{-1}\left(A^{-1}B - A^{-1}B(D - CA^{-1}B)^{-1}(D - CA^{-1}B)\right)D^\perp = 0.
\end{aligned}$$

This implies that the columns of $BD^\perp$ are eigenvectors of M corresponding to zero eigenvalues. Therefore, M has at least $m - p$ zero eigenvalues, that is $\det(\lambda I - M) = \lambda^{(m-p)}q(\lambda)$. Consequently, M has exactly $m - p$ zero eigenvalues.

Proof of sufficiency: Assuming that $D - CA^{-1}B > 0$, and M has no real negative eigenvalue and has exactly $m - p$ zero eigenvalues implies that $\det(\lambda I - M) = \lambda^{(m-p)}q(\lambda)$, with $q(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}^-$. Consequently, $\det(\omega^2 I + M) \neq 0$ for all $\omega \neq 0$. Due to continuity of the determinant with respect to ω everywhere we have $\det(\omega^2 I + M) > 0$ for all $\omega \neq 0$. With the same argument we can state that (25) holds, and also $\det(\omega^2 I + \bar{A}^2) > 0$ as $\bar{A}$ is Hurwitz. Now, from the computation (20) and (23), and positive definiteness of $\bar{D} = D - CA^{-1}B$ it immediately follows that (18) and (19) are valid. Then, with changing the variable ω to $-\frac{1}{\omega}$ in (18) and (19), we conclude

$$\det\left(G(j\omega) + G^T(-j\omega)\right) > 0, \quad \forall\,\omega \in \mathbb{R}, \tag{26}$$

and (8) holds. Now, we demonstrate that (26) implies that $G(j\omega) + G^T(-j\omega) > 0$. In fact, $G(j\omega) + G^T(-j\omega)$ is a Hermitian matrix, implying that its eigenvalues are all real and at $\omega = 0$ is positive definite. Therefore, if for some frequency it fails to be positive definite, then there must exist an $\omega = \omega_1 \in \mathbb{R}$ such that at least one eigenvalue of this matrix is equal to zero, that is, $\det\left(G(j\omega_1) + G^T(-j\omega_1)\right) = 0$. This is obviously in contradiction with (26). Now, the requirements of Lemma 2 are fulfilled and $G(s)$ to be SPR is inferred.

4. Main results

We now present the main result of this note concerning the existence of a strong or a weak Lyapunov function for a certain class of switched linear systems.

Strong common quadratic Lyapunov function: Consider the switched linear system (3) with appropriate dimensions for A, B, C, and D as introduced in Section 1. Both matrices A and $A - BD^{-1}C$ are assumed to be Hurwitz. Exponential stability of the switched system (3) is guaranteed if there exists a $P = P^T > 0$ such that

$$A^T P + PA < 0, \tag{27}$$

$$(A - BD^{-1}C)^T P + P(A - BD^{-1}C) < 0. \tag{28}$$

We want to explore under which condition such a strong CQLF $V(x) = x^T Px$ exists.

Theorem 1. *Consider two Hurwitz matrices A and $A - BD^{-1}C$ with (A, B) controllable and (A, C) observable, satisfying $D = D^T > 0$ and (12). Then the switched system (3) is quadratically stable if and only if $A(A - BD^{-1}C)$ has no real negative eigenvalue.*

Proof of necessity: Suppose that $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$ share a strong CQLF. Then, by pre-multiplying the inequality $A^T P + PA < 0$ by the nonsingular matrix A^{-T} and post-multiplying it by A^{-1}, we get $A^{-T}P + PA^{-1} < 0$. This means that $\dot{x} = A^{-1}x$, $\dot{x} = (A - BD^{-1}C)x$, and consequently the family of systems $\dot{x} = (\omega^2 A^{-1} + (A - BD^{-1}C))x$ share the same CQLF $V(x) = x^T Px$ for all $\omega \in \mathbb{R}$;

$$\begin{aligned}
&\left[\omega^2 A^{-1} + (A - BD^{-1}C)\right]^T P \\
&\quad + P\left[\omega^2 A^{-1} + (A - BD^{-1}C)\right] < 0;
\end{aligned} \tag{29}$$

as in Shorten and Narendra (2003). Hence, it follows from Lyapunov's second theorem that the matrix $\omega^2 A^{-1} + (A - BD^{-1}C)$ is Hurwitz for all $\omega \in \mathbb{R}$ and thus is non-singular, that is

$$\begin{aligned}
&\det\left(\omega^2 A^{-1} + (A - BD^{-1}C)\right) \neq 0 \\
&\quad \Rightarrow \det(A^{-1})\ \det\left(\omega^2 I + A(A - BD^{-1}C)\right) \neq 0.
\end{aligned}$$

As A^{-1} is Hurwitz, the latter implies that

$$\det\left(\omega^2 I + A(A - BD^{-1}C)\right) \neq 0, \tag{30}$$

or equivalently $A(A - BD^{-1}C)$ has no real negative eigenvalue. Note that for the proof of necessity the symmetry conditions given in (12) are not demanded.

Proof of sufficiency: Recall that $A(A - BD^{-1}C)$ having no real negative eigenvalue and the symmetry conditions (12) and $D = D^T > 0$ imply that $G(s) = C(sI - A)^{-1}B + D$ is SPR. Then, referring to the KYP lemma, there must exist a matrix $P = P^T > 0$, a scalar $\alpha > 0$, and matrices L and W satisfying Eq. (9)–(11). Moreover, we show that the function $V(x) = x^T Px$ is a strong Lyapunov function for both $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$. In fact, $A^T P + PA < 0$ immediately follows from Eq. (9). In the following we show that also $(A - BD^{-1}C)^T P + P(A - BD^{-1}C) < 0$ holds. To see that, we have

$$\begin{aligned}(A - BD^{-1}C)^T P + P(A - BD^{-1}C) \\ = -\alpha P - L^T L - C^T D^{-1}(C - W^T L) - (C - W^T L)^T D^{-1} C \\ = -\alpha P - (L - WD^{-1}C)^T (L - WD^{-1}C) < 0, \end{aligned} \tag{31}$$

where we used the identities (9), (10), and (11); see also Kouhi et al. (2013).

Weak common quadratic Lyapunov function: Consider again the switched linear system (3) where A, B, C, and D have appropriate dimensions as stated in Section 1. In this part, we assume A is Hurwitz but $A - BD^{-1}C$ has all eigenvalues with negative real parts except some eigenvalues equal to zero. We want to study under which conditions the switched system (3) possesses a weak CQLF in the sense that there exists a $P = P^T > 0$ such that

$$A^T P + PA < 0, \tag{32}$$

$$(A - BD^{-1}C)^T P + P(A - BD^{-1}C) \leq 0. \tag{33}$$

Referring to the discussion in Shorten et al. (2009), for this problem we cannot directly exploit the idea of using Theorem 1 due to the existence of the equality sign in (33). In the Appendix, we demonstrate that if (32) and (33) hold then $A - BD^{-1}C$ can have at most m eigenvalues equal to zero with no generalized eigenvector. In addition, $A - BD^{-1}C$ cannot have any eigenvalue on the $j\omega$ axis when $G(j\omega) = C(j\omega I - A)^{-1}B + D$ is symmetric and $\omega \in \mathbb{R} \setminus \{0\}$.

Theorem 2. *Assume A is Hurwitz and all eigenvalues of $A - BD^{-1}C$ have negative real parts or are zero. Furthermore, assume that the zero eigenvalue has a multiplicity of $m - p$, and associated with the zero eigenvalue is a full set of $m - p$ linearly independent eigenvectors. Suppose also that $D = D^T > 0$, (12) holds, (A, B) is controllable, and (A, C) is observable. Then, the switched system (3) is weakly quadratically stable if and only if $A(A - BD^{-1}C)$ has no real negative eigenvalue and has exactly $m - p$ eigenvalues equal to zero.*

Proof of sufficiency: Recall that Lemma 4 implies that the symmetry conditions (12) and $D = D^T$ are sufficient for the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ to be symmetric. The remainder of the proof of sufficiency consists of two parts. In Part A, we prove that $G(0) \geq 0$ and $\text{rank}(G(0)) = p$. Afterwards, in Part B we use $G(\frac{1}{s})$ in combination with the KYP lemma to complete the proof of sufficiency.

Part A: Following the proof of sufficiency in Lemma 6, $A(A - BD^{-1}C)$ having no real negative eigenvalues implies that $\det(\omega^2 I + A(A - BD^{-1}C)) > 0$ for all $\omega \in \mathbb{R} \setminus \{0\}$. Therefore, recalling (14) we can deduce $\det(D - C(\omega^2 I + A^2)^{-1}AB) > 0$ for all $\omega \in \mathbb{R} \setminus \{0\}$. Hence, analogous to the proof of sufficiency in Theorem 1, we can argue that for all $\omega \in \mathbb{R} \setminus \{0\}$

$$\frac{1}{2}\{G(j\omega) + G^T(-j\omega)\} = D - C(\omega^2 I + A^2)^{-1}AB > 0.$$

Now, by continuity of the eigenvalues with respect to ω around zero, the matrix $G(0) = \lim_{\omega \to 0} \frac{1}{2}\{G(j\omega) + G^T(-j\omega)\} = D - CA^{-1}B$ cannot have any eigenvalue with negative real part, that is $G(0) = D - CA^{-1}B \geq 0$. Next, we prove that $G(0)$ has exactly $m - p$ eigenvalues equal to zero. To this end, consider the equality $(A - BD^{-1}C)A^{-1}B = BD^{-1}(D - CA^{-1}B)$, then exploiting the Sylvester rank inequality (Horn & Johnson, 1990), we have

$$\begin{aligned}\text{rank}\left(BD^{-1}(D - CA^{-1}B)\right) = \text{rank}\left((A - BD^{-1}C)A^{-1}B\right) \\ \geq \text{rank}(A - BD^{-1}C) + \text{rank}(A^{-1}B) - n \\ = [n - (m - p)] + m - n = p.\end{aligned}$$

As $\text{rank}(B) = m$ and D is invertible, we have $\text{rank}(BD^{-1}) = m \geq p$ and therefore $\text{rank}(D - CA^{-1}B) \geq p$. On the other hand, $A - BD^{-1}C$ has exactly $m - p$ eigenvalues equal to zero with a full set of eigenvectors, so there exists a matrix $\mathcal{W}^T \in \mathbb{R}^{(m-p)\times n}$ consisting of the left eigenvectors of $A - BD^{-1}C$ corresponding to the zero eigenvalues, that is $\mathcal{W}^T(A - BD^{-1}C) = 0$. Note that the rows of $\mathcal{W}^T BD^{-1}$ are linearly independent since $\mathcal{W}^T A$ has full row rank and $\mathcal{W}^T A = \mathcal{W}^T BD^{-1}C$. It then follows from $\mathcal{W}^T(A - BD^{-1}C)A^{-1}B = 0$ that $\mathcal{W}^T BD^{-1}(D - CA^{-1}B) = 0$. Therefore, the rows of $\mathcal{W}^T BD^{-1} \in \mathbb{R}^{(m-p)\times m}$ are indeed left eigenvectors of $G(0) = D - CA^{-1}B$ corresponding to the zero eigenvalues. Note also that the symmetry of $G(0)$ excludes the possibility of the existence of any generalized eigenvector for $G(0)$. As a result, since $\text{rank}(G(0)) \geq p$ and $G(0)$ has at least $m - p$ left eigenvectors corresponding to zero eigenvalues, we have $\text{rank}(D - CA^{-1}B) = p$, indicating that this matrix has exactly $m - p$ eigenvalues equal to zero.

Part B: Define the system $\bar{G}(s) := G(\frac{1}{s}) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$, with the state space realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ as $\bar{A} = A^{-1}$, $\bar{B} = -A^{-1}B$, $\bar{C} = CA^{-1}$, and $\bar{D} = D - CA^{-1}B$. Consequently, symmetry of $\bar{G}(s)$ follows from symmetry of $G(s)$. Furthermore, $(\bar{A}, \bar{B})$ and $(\bar{A}, \bar{C})$ are controllable and observable, respectively, and $\bar{D} \geq 0$ has rank p. As $\bar{A}$ is Hurwitz, recall that Lemma 7 implies that $A(A - BD^{-1}C)$ having no real negative and exactly $m - p$ zero eigenvalues is equivalent to $\bar{G}(s)$ being SPR. Then, the KYP lemma implies that there exist a scalar $\alpha > 0$ and matrices $P = P^T > 0$, L, and W such that

$$\bar{A}^T P + P\bar{A} = -L^T L - \alpha P \tag{34}$$

$$\bar{B}^T P + W^T L = \bar{C} \tag{35}$$

$$\bar{D} + \bar{D}^T = W^T W. \tag{36}$$

Eq. (34) ensures that $A^{-T}P + PA^{-1} < 0$. By pre- and post-multiplying of this inequality by A^T and A^{-T}, respectively, we get $A^T P + PA < 0$, which verifies that (32) holds. Next, we show that (33) also holds. Let us initiate the proof by showing

$$\begin{bmatrix} A^{-T}P + PA^{-1} & P\bar{B} - \bar{C}^T \\ \bar{B}^T P - \bar{C} & -2\bar{D} \end{bmatrix} \leq 0. \tag{37}$$

Note that $\bar{D} \geq 0$, and we shall demonstrate that the Schur complement of the matrix in (37) is less or equal to zero, in other words

$$\bar{\mathcal{S}} = -2\bar{D} - (\bar{B}^T P - \bar{C})(A^{-T}P + PA^{-1})^{-1}(P\bar{B} - \bar{C}^T) \leq 0.$$

Using Eqs. (34)–(36), we get

$$\begin{aligned}\bar{\mathcal{S}} = -2\bar{D} - (\bar{B}^T P - \bar{C})(A^{-T}P + PA^{-1})^{-1}(P\bar{B} - \bar{C}^T) \\ = -W^T W + W^T L(\alpha P + L^T L)^{-1} L^T W \\ = -W^T \left[I - L(\alpha P + L^T L)^{-1} L^T\right] W.\end{aligned} \tag{38}$$

Notice in (38), $I - L(\alpha P + L^T L)^{-1}L^T > 0$ since it is the Schur complement of the positive definite matrix

$$\begin{bmatrix} \alpha P + L^T L & L^T \\ L & I \end{bmatrix}.$$

Hence, $\bar{\mathcal{S}} < 0$. Now, let us reformulate (37) as

$$\begin{bmatrix} A^{-1} & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix} \leq 0. \tag{39}$$

Since $\bar{D} - \bar{C}A\bar{B} = D$, we have

$$\begin{bmatrix} A^{-1} & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix}^{-1} = \begin{bmatrix} A - A\bar{B}(-\bar{D} + \bar{C}A\bar{B})^{-1}\bar{C}A & -A\bar{B}(-\bar{D} + \bar{C}A\bar{B})^{-1} \\ (-\bar{D} + \bar{C}A\bar{B})^{-1}\bar{C}A & (-\bar{D} + \bar{C}A\bar{B})^{-1} \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & -BD^{-1} \\ -D^{-1}C & -D^{-1} \end{bmatrix}. \quad (40)$$

By pre-multiplying the inequality (39) by the transpose of the above matrix in (40) and by post-multiplying it by the above matrix itself, we end up with

$$\begin{bmatrix} (A - BD^{-1}C)^T P + P(A - BD^{-1}C) & -(PB + C^T)D^{-1} \\ -D^{-1}(B^T P + C) & -2D^{-1} \end{bmatrix} \leq 0.$$

From this, it immediately follows that $(A - BD^{-1}C)^T P + P(A - BD^{-1}C) \leq 0$.

Proof of necessity: A proof that $A(A - BD^{-1}C)$ does not have any real negative eigenvalue is similar to the proof of necessity given for Theorem 1.

The last part of the proof is concerned with proving that the product AA_2, with $A_2 = A - BD^{-1}C$, has exactly $m - p$ eigenvalues equal to zero. To show this holds, first note that AA_2 has exactly $m - p$ zero eigenvalues with a full set of eigenvectors. This fact can be derived by considering that $\text{rank}(AA_2) = \text{rank}(A_2)$. Therefore, if AA_2 has more than $m - p$ zero eigenvalues, then it must contain at least one generalized eigenvector, say $v_2 \in \mathbb{R}^n$ satisfying $AA_2 v_2 = v_1$ and $AA_2 v_1 = 0$. This implies that

$$A_2 v_2 = A^{-1} v_1, \quad \text{and} \quad A_2 v_1 = 0. \quad (41)$$

Moreover, it can be inferred from the inequality $A^{-T}P + PA^{-1} < 0$ that the function $f(x) = x^T(A^{-T}P + PA^{-1})x$ is always negative for all non zero $x \in \mathbb{R}^n$, in particular for $x = v_1$. Recalling (41), we have

$$v_1^T(A^{-T}P + PA^{-1})v_1 < 0 \Rightarrow v_2^T A_2^T P v_1 + v_1^T P A_2 v_2 < 0. \quad (42)$$

On the other hand, with regard to (33) the inequality $g(x) = x^T(A_2^T P + PA_2)x \leq 0$ must hold for all $x \in \mathbb{R}^n$. Now, choosing $x = \beta v_1 + v_2$ with $\beta \in \mathbb{R}$ as a parameter and considering $A_2 v_1 = 0$, we have

$$(\beta v_1 + v_2)^T (A_2^T P + PA_2)(\beta v_1 + v_2) \leq 0$$
$$\Rightarrow \beta(v_2^T A_2^T P v_1 + v_1^T P A_2 v_2) + v_2^T(A_2^T P + PA_2)v_2 \leq 0. \quad (43)$$

As the inequalities $e = v_2^T A_2^T P v_1 + v_1^T P A_2 v_2 < 0$ and $g(v_2) = v_2^T(A_2^T P + PA_2)v_2 \leq 0$ hold, there exists always a $\beta \in \mathbb{R}$ which makes the expression $\beta e + g(v_2)$ positive, for instance, one can choose $\beta < -g(v_2)/e$. However, this is evidently in contradiction with (43).

5. Examples

We now present two examples to illustrate our results.

Example 1. Consider two matrices

$$A_1 = \begin{bmatrix} -1 & 0 \\ 5 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 0 & -5 \end{bmatrix}.$$

A_1 is Hurwitz and A_2 has all eigenvalues in the open left half plane except one eigenvalue at zero. In addition, $A_1 - A_2$ has rank two and with

$$B = \begin{bmatrix} 1 & 2 \\ 10 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.2 \\ -1 & 0.4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

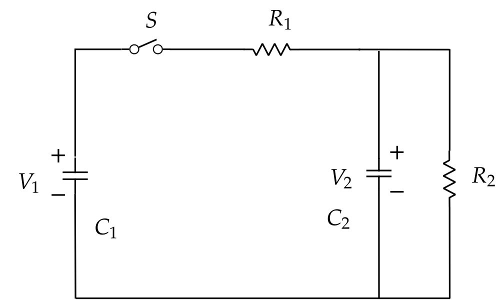

Fig. 1. An electrical circuit.

we have $A_1 - A_2 = BD^{-1}C$. Moreover, with $A = A_1$, the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is symmetric

$$G(s) = \frac{1}{(s+1)^2} \begin{bmatrix} s^2 + 5s + 5 & 3s + 5 \\ 3s + 5 & s^2 + 2s + 5 \end{bmatrix}.$$

Note that (A, B) and (A, C) are controllable and observable, respectively. Both eigenvalues of the matrix

$$A_1 A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

are zero. Hence, by Theorem 2 there is no weak CQLF for the switched system associated with A_1 and A_2. Now, let us change A_1 to the Hurwitz matrix

$$A_1 = \begin{bmatrix} -0.9 & -0.1 \\ 4.5 & -1.4 \end{bmatrix}.$$

Again, in this case $A_1 - A_2$ has rank two, and with

$$B = \begin{bmatrix} 0.9 & 1.8 \\ 9 & 4.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.2 \\ -1 & 0.4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have $A_1 - A_2 = BD^{-1}C$. Moreover, with $A = A_1$, the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is symmetric

$$G(s) = \frac{1}{s^2 + 2.3s + 1.71} \begin{bmatrix} s^2 + 5s + 4.5 & 2.7s + 4.5 \\ 2.7s + 4.5 & s^2 + 2.3s + 4.5 \end{bmatrix}.$$

Note that (A, B) and (A, C) are controllable and observable, respectively. Moreover, the eigenvalues of the matrix

$$A_1 A_2 = \begin{bmatrix} 0 & 1.4 \\ 0 & 2.5 \end{bmatrix},$$

are 0 and 2.5, implying that no real negative eigenvalue for the product of the two matrices exists. Hence, according to Theorem 2 there is a weak CQLF for the switched system associated with the two matrices.

Example 2. Consider the switched electrical circuit illustrated in Fig. 1. The variables V_1 and V_2 are voltages of two capacitors with capacities C_1 and C_2, respectively. Let $x := [V_1\ V_2]^T$ indicate the vector of the system states. Depending on the status of the switch S, the system model can be represented by a switched linear system in the form of (1) with the data

$$A_1 = \begin{bmatrix} \frac{-1}{R_1 C_1} & \frac{1}{R_1 C_1} \\ \frac{1}{R_1 C_2} & \frac{-(R_1 + R_2)}{R_1 R_2} \frac{1}{C_2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{R_2 C_2} \end{bmatrix}. \quad (44)$$

In (44) A_1 refers to the situation when S is closed and A_2 refers to the condition when S is open. Note that A_1 is Hurwitz and A_2 is singular including exactly one zero eigenvalue and one real negative eigenvalue. In addition, A_1 and A_2 have rank one difference. Defining $A := A_1$, $B := [-1/C_1\ 1/C_2]^T$, $C :=$

$[1/R_1\ -1/R_1]$, and $D := I$, we have $A_2 = A - BD^{-1}C$ and the transfer function $G(s) = C(sI - A)^{-1}B + D$ is symmetric. Furthermore, (A, B) and (A, C) are controllable and observable, respectively. Now, consider that the matrix

$$A_1A_2 = \begin{bmatrix} 0 & \dfrac{-1}{R_1R_2C_1C_2} \\ 0 & \dfrac{(R_1+R_2)}{R_1R_2^2C_2^2} \end{bmatrix},$$

has no real negative eigenvalue and has exactly one zero eigenvalue. Hence, according to Theorem 2 this electrical circuit is weakly quadratically stable, implying that the voltages of both capacitors remain bounded in spite of all possible switching events and each initial condition.

6. Conclusions

In this paper we studied the stability properties of a class of switched linear systems whose system matrices satisfy certain symmetry conditions.

Appendix

Part 1: In this part we give a proof for Lemma 1.

Proof of necessity: Let $A = A_1$, suppose that real matrices B, C, and $D = D^T > 0$ with appropriate dimensions exist such that $G(s) = C(sI - A)^{-1}B + D$ is symmetric, the equality $A_2 = A - BD^{-1}C$ holds, and (A, B, C, D) is a minimal realization of $G(s)$. Then, there must exist a symmetric invertible matrix S (not necessarily positive definite) which satisfies the relationships

$$A^TS - SA = 0, \qquad B^TS = C; \tag{45}$$

see Kailath (1980) and Knockaert, Ferranti, and Dhaene (2013). Substituting C from (45) into the equation $A_2 = A - BD^{-1}C$, we have $A_2 = A - BD^{-1}B^TS$, consequently $A_2S^{-1} = AS^{-1} - BD^{-1}B^T$. From (45) one can deduce that $AS^{-1} = S^{-1}A^T$. Therefore, the matrix $AS^{-1} - BD^{-1}B^T = A_2S^{-1}$ is symmetric as well, that is $A_2S^{-1} - S^{-1}A_2^T = 0$. Defining $Y := S^{-1}$, both $A_1 = A$ and A_2 satisfy the following Sylvester equations

$$A_1Y - YA_1^T = 0, \qquad A_2Y - YA_2^T = 0. \tag{46}$$

For finding a solution of the Sylvester equations, utilizing the Kronecker product notation and the vectorization operator, we can reformulate (46) in the form of

$$(I \otimes A_1 - A_1 \otimes I)\ \mathrm{vec}Y = 0$$
$$(I \otimes A_2 - A_2 \otimes I)\ \mathrm{vec}Y = 0; \tag{47}$$

see Horn and Johnson (1990). It is obvious that (47) implies $E_1 = (I \otimes A_1 - A_1 \otimes I)$ and $E_2 = (I \otimes A_2 - A_2 \otimes I)$ share a common eigenvector $\mathrm{vec}(Y)$ corresponding to a zero eigenvalue, where Y is symmetric and invertible, and $(A_1 - A_2)Y = BD^{-1}B^T$ is positive semi-definite. Note that each of E_1 and E_2 have at least n zero eigenvalues; see Horn and Johnson (1990).

Proof of sufficiency: Suppose E_1 and E_2 share a common eigenvector $\mathrm{vec}(Y)$ corresponding to a zero eigenvalue where Y is symmetric and invertible (Note that the assumption for Y to be symmetric is not restrictive because if X is a solution of $A_1X - XA_1^T = 0$ then $(XA_1^T)^T = (A_1X)^T$, and from $A_1(X + X^T) = XA_1^T + (A_1X)^T = XA_1^T + X^TA_1^T = (X + X^T)A_1^T$, we realize that the symmetric matrix $(X + X^T)$ is also a solution to the equation $A_1X - XA_1^T = 0$.) Then, Y satisfies (46). This implies that the matrix $(A_1 - A_2)Y$ is symmetric. Suppose $\mathrm{rank}\,((A_1 - A_2)Y) = m$. As $(A_1 - A_2)Y$ is symmetric and by the assumption of the lemma is positive semi-definite, there exists a matrix $B \in \mathbb{R}^{n\times m}$ such that $(A_1 - A_2)Y = BB^T$. Defining $A := A_1$, $S := Y^{-1}$, $C := B^TS$, and $D := I$, we have $A_1 = A$, $A_2 = A - BD^{-1}C$, and $G(s) = C(sI - A)^{-1}B + D$ is symmetric.

Part 2: In this part, we first prove that if (32) and (33) hold and $G(s) = C(sI - A)^{-1}B + D$ is symmetric, if A is Hurwitz and D is positive definite, then $A - BD^{-1}C$ cannot have any eigenvalue on the $j\omega$ axis for $\omega \in \mathbb{R} \setminus \{0\}$. Let us assume there exists an eigenvalue $j\omega_1$, with finite $\omega_1 \neq 0$. Note that A being Hurwitz implies $\det(j\omega_1 I - A) \neq 0$, and $D > 0$ implies $\det(D^{-1}) \neq 0$. Then,

$$\begin{aligned} &\det\left(j\omega_1 I - (A - BD^{-1}C)\right) = 0 \\ &\Rightarrow \det(j\omega_1 I - A)\det\left(I + (j\omega_1 I - A)^{-1}BD^{-1}C\right) = 0 \\ &\Rightarrow \det(D^{-1})\det\left(D + C(j\omega_1 I - A)^{-1}B\right) = 0 \\ &\Rightarrow \det\left(D + C(j\omega_1 I - A)^{-1}B\right) = 0. \end{aligned} \tag{48}$$

This implies that the matrix $G(j\omega_1)$ is singular. Let us denote the complex eigenvector corresponding to the eigenvalue 0 of $G(j\omega_1)$ by $v \in \mathbb{C}^m$. Similarly we can show that the matrix $G(-j\omega_1) = G^T(-j\omega_1) = G^*(j\omega_1)$ is also singular and has a left eigenvector v^* corresponding to the eigenvalue 0. Thus, recalling Lemma 5, the following holds

$$\begin{aligned} \frac{1}{2}v^*[G(j\omega_1) + G^T(-j\omega_1)]v &= v^*[D - C(\omega_1^2 I + A^2)^{-1}AB]v \\ &= 0. \end{aligned} \tag{49}$$

On the other hand, as discussed in the proof of the main results, a necessary condition for (32) and (33) to hold is that the matrix $\omega^2 I + A(A - BD^{-1}C)$ for $\omega \neq 0$ is nonsingular, that is

$$\begin{aligned} &\det\left(\omega^2 I + A(A - BD^{-1}C)\right) > 0 \\ &\Rightarrow \det\left(\frac{1}{2}\{G(j\omega) + G^T(-j\omega)\}\right) > 0, \end{aligned} \tag{50}$$

which by continuity of the determinant with respect to ω indicates that $G(j\omega) + G^T(-j\omega)$ is positive definite for all $\omega \in \mathbb{R} \setminus \{0\}$; in particular for $\omega = \omega_1$. However, this is the contradiction with (49).

Finally, we prove $A - BD^{-1}C$ can have at most m eigenvalues equal to 0. This proof is similar to the proof given in Shorten et al. (2009). Since inequality (33) holds, the system $\dot{x} = (A - BD^{-1}C)x$ is stable, that is the algebraic multiplicity and geometric multiplicity of the zero eigenvalues are the same. Now, suppose that we have more than m linearly independent eigenvectors corresponding to zero eigenvalues. Let us define $Q_1 = A^TP + PA$ and $Q_2 = (A - BD^{-1}C)^TP + P(A - BD^{-1}C)$, and consider the functions $g_1(x) = x^TQ_1x$ and $g_2(x) = x^TQ_2x$. Obviously, at least for $n - m$ linearly independent vectors $v_i \in \mathbb{R}^n$, $i = 1, \ldots, n - m$; we have $g_1(v_i) = g_2(v_i)$, for instance, for those v_i's that satisfy $v_i^TPB = 0$. Suppose there are more than m linearly independent vectors x such that $g_2(x)$ is zero, then the two spaces $\{x : g_2(x) = 0\}$ and $\{x : g_2(x) = g_1(x)\}$ intersect at least at one non-zero point, implying that there exists a vector x such that $g_1(x) = g_2(x) = 0$. The condition $g_1(x) = 0$ is in contradiction with the choice of P in (32).

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