Stock Reduction for Timed Event Graphs Based on Output Feedback

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Abstract: In timed event graphs (TEG), stock represents the number of tokens present in the system. It is very similar to work-in-process for manufacturing systems. In general, when choosing a feedback controller, a compromise is sought between fastness of the system and stock size. The classical choice in the $(\max, +)$ literature consists in reducing the stock as much as possible without delaying the output. In this paper, the constraint is weakened: the response of the controlled system to a specific, predefined reference input w must be as fast as the one of the uncontrolled system, but may be slower for other inputs. In return, lower stock is expected. After formally defining this new controller, its performance in terms of stock is compared with the one of the classical controller. This last part relies heavily on second order theory for TEG.

Keywords: Timed Event Graphs, Max Plus Algebra, Idempotent Semiring, Control, Linear Systems, Output Feedback, Work-in-Process

1. INTRODUCTION

Timed event graphs (TEG) constitute a subclass of timed Petri nets where each place has exactly one upstream and one downstream transition and all arcs have weight 1. It is well known that the timed/event behavior of a TEG, under the earliest functioning rule (*i.e.,* a transition is fired as soon as it is enabled), can be expressed by linear relations over some dioids (Baccelli et al. (1992)). Applications of these models are numerous within the framework of production management. In this field, an interesting problem is stock reduction. Stock represents the number of tokens present in the system. It is a non-linear function in dioids and leads to a second order theory for TEG (MaxPlus (1991)). To take into account perturbations (unexpected failure, ...) in stock reduction, a feedback is considered. When choosing a feedback controller, a compromise is sought between fastness of the system and stock reduction: the greater the feedback controller is, the slower the system is, and the lower the stock is. In the (max,+) literature, an *optimal output feedback controller* for TEG exists and is proposed in (Cottenceau et al. (1999)) and (Cottenceau et al. (2001)). It is the greatest output feedback controller preserving the transfer function of the uncontrolled system: the response of the controlled system to every input u is as fast as the one of the uncontrolled system. Subject to the restriction that the system is never slowed down, stock reduction is maximal. However, for some applications, it could be interesting to obtain a lower stock, even though this implies slowing the system down. The approach presented in this paper is based on an output feedback controller, which requires that the response of the controlled system to a specific, predefined reference input w is as fast as the one of the uncontrolled system, but may be slower for other inputs. The choice of the reference input w could be based on an estimation of the input (expected supply of raw materials, ...).

In § 2, necessary algebraic tools are given. TEG modeling over the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is recalled in § 3. In § 4, second order theory for TEG is presented. In § 5, a new output feedback controller is defined and is compared, with respect to stock reduction, to the one defined in (Cottenceau et al. (1999)).

2. ALGEBRAIC PRELIMINARIES

2.1 Dioid Theory

The following is a short summary of basic results from dioid theory. The reader is invited to consult (Baccelli et al. (1992)) for more details.

Definition 1. (Dioid, Complete Dioid). A dioid $\mathcal{D}$ is a set endowed with two internal operations denoted $\oplus$ (addition) and $\otimes$ (multiplication, often denoted by juxtaposition), both associative and both having a neutral element denoted ε and e respectively. Moreover, $\oplus$ is commutative and idempotent ($\forall a \in \mathcal{D}, a \oplus a = a$), $\otimes$ is distributive with respect to $\oplus$, and ε is absorbing for $\otimes$ ($\forall a \in \mathcal{D}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$).

A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. The sum of all its elements is denoted $\top$.

Definition 2. (Order relation). A dioid is endowed with a partial order denoted $\succeq$ and defined by the following equivalence: $a \succeq b \Leftrightarrow a = a \oplus b$. $a \oplus b$ is the least upper bound of $\{a, b\}$.

Remark 3. A complete dioid can be equipped with an additional internal operation $\wedge$. $a \wedge b$ is defined as the greatest lower bound of $\{a, b\}$. $\wedge$ is associative, commutative, idempotent and has a neutral element $\top$.

Example 4. A well-known complete dioid is the $(\min, +)$-algebra: $\mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with min as addition and $+$ as multiplication. ε is equal to $+\infty$ and $\top$ to $-\infty$. The associated order relation is dual to the classical order relation on $\mathbb{Z}$.

By analogy with linear algebra, operations on matrices with entries in a complete dioid $\mathcal{D}$ are defined. For all $A, B \in \mathcal{D}^{n\times p}$ and $C \in \mathcal{D}^{p\times m}$:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \quad (1)$$

$$(A \wedge B)_{ij} = A_{ij} \wedge B_{ij} \quad (2)$$

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{p} A_{ik} C_{kj} \quad (3)$$

Endowed with the previous operations, the set of square matrices with entries in a complete dioid is also a complete dioid.

Definition 5. (Subdioid). Let $\mathcal{D}$ be a dioid and $\mathcal{D}_{sub} \subseteq \mathcal{D}$. $\mathcal{D}_{sub}$ is said to be a subdioid of $\mathcal{D}$ if $\varepsilon, e \in \mathcal{D}_{sub}$ and $\mathcal{D}_{sub}$ is closed for $\oplus$ and $\otimes$.

The following theorem plays a fundamental role for the study of TEG behavior under the earliest functioning rule.

Theorem 6. (Kleene star theorem). The implicit equation $x = ax \oplus b$ defined over a complete dioid admits $x = a^*b$ as least solution with $a^* = \bigoplus_{i\geqslant 0} a^i$ (Kleene star).

Remark 7. Interesting properties of the Kleene star are $a^*a^* = a^*$ and $a^* \succeq e$.

2.2 Residuation Theory

In ordered sets, like dioids, equations $f(x) = b$ may have either no solution, one solution, or multiple solutions. In order to give always a unique answer to the problem of mapping inversion, residuation theory (Blyth and Janowitz (1972)) provides, under some assumptions, either the greatest solution (in accordance with the considered order) to the inequality $f(x) \preceq b$ or the least solution to the inequality $f(x) \succeq b$.

Definition 8. (Isotone mapping). A mapping f defined over ordered sets is said to be isotone if $a \preceq b \Rightarrow f(a) \preceq f(b)$.

Definition 9. (Residuation). Let $f : \mathcal{E} \to \mathcal{F}$, with $(\mathcal{E}, \preceq)$ and $(\mathcal{F}, \preceq)$ ordered sets. An isotone mapping f is said to be residuated if for all $y \in \mathcal{F}$, the least upper bound of the subset $\{x \in \mathcal{E} | f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f.

The following theorem gives a very handy characterization of residuated mappings when the considered ordered sets are complete dioids.

Theorem 10. (Baccelli et al. (1992)). Let $f : \mathcal{D} \to \mathcal{E}$ be an isotone mapping defined over complete dioids. Mapping f is residuated if and only if $f(\varepsilon) = \varepsilon$ and, $\forall A \subseteq \mathcal{D}$, $f(\bigoplus_{x\in A} x) = \bigoplus_{x\in A} f(x)$.

Corollary 11. Let $L_a : x \mapsto a \otimes x$ (left-product by a) and $R_a : x \mapsto x \otimes a$ (right-product by a) be defined on a complete dioid. Mappings L_a and R_a are both residuated. Their residuals will be denoted respectively $L_a^\sharp(x) = a \backslash\!\!\circ\, x$ (left-division by a) and $R_a^\sharp(x) = x \,/\!\!\circ\, a$ (right-division by a).

Remark 12. In the $(\min, +)$-algebra, $a \,/\!\!\circ\, b$ is equal to $a - b$, where subtraction is extended to $+\infty$ and $-\infty$, using the properties of ε and $\top$.

For matrices with entries in a complete dioid $\mathcal{D}$, the right- (or left-)product is residuated. Besides, the calculation of the residual can be derived from residuals in $\mathcal{D}$ as explained in the following proposition.

Proposition 13. (Baccelli et al. (1992)). Consider a complete dioid $\mathcal{D}$, $A \in \mathcal{D}^{n\times m}$, $B \in \mathcal{D}^{n\times p}$, and $C \in \mathcal{D}^{p\times m}$, then:

- $B \backslash\!\!\circ\, A \in \mathcal{D}^{p\times m}$ with $(B \backslash\!\!\circ\, A)_{ij} = \bigwedge_{k=1}^{n} B_{ki} \backslash\!\!\circ\, A_{kj}$
- $A \,/\!\!\circ\, C \in \mathcal{D}^{n\times p}$ with $(A \,/\!\!\circ\, C)_{ij} = \bigwedge_{k=1}^{m} A_{ik} \,/\!\!\circ\, C_{jk}$

In the following other interesting results from residuation theory are recalled.

Theorem 14. (Cottenceau et al. (2001)). Let $\mathcal{D}$ be a complete dioid and $A \in \mathcal{D}^{p\times n}$. Then, $A \backslash\!\!\circ\, A \in \mathcal{D}^{n\times n}$ and $A \,/\!\!\circ\, A \in \mathcal{D}^{p\times p}$. Moreover, $A \,/\!\!\circ\, A = (A \,/\!\!\circ\, A)^*$ and $A \backslash\!\!\circ\, A = (A \backslash\!\!\circ\, A)^*$.

Proposition 15. (Cottenceau et al. (2001)). In a complete dioid $\mathcal{D}$, the greatest solution to inequality $x^* \preceq a^*$ is $x = a^*$.

3. TEG DESCRIPTION

The input-output behavior of a TEG may be represented by a transfer relation in some particular dioids. Hereafter, TEG behavior is essentially described in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. This dioid is briefly presented below, but a complete description is available in (Cohen et al. (1989)) or (Baccelli et al. (1992)).

The dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is based on $\mathbb{B}[\![\gamma, \delta]\!]$, the set of formal power series in two variables (γ, δ) with Boolean coefficients and with exponents in $\mathbb{Z} \cup \{-\infty, +\infty\}$. The timed/event behavior of a TEG complies with structural properties: the number of events is increasing with respect to the time or equivalently the firing time is increasing with respect to the events. Then, a filtering of $\mathbb{B}[\![\gamma, \delta]\!]$ is done to take into account the previous properties: $\mathbb{B}[\![\gamma, \delta]\!]$ is equipped with a congruence relation (*i.e.,* an equivalence relation compatible with $\oplus$ and $\otimes$): $x\mathcal{R}y \Leftrightarrow \gamma^*(\delta^{-1})^*x = \gamma^*(\delta^{-1})^*y$. The dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is defined as the quotient dioid of $\mathbb{B}[\![\gamma, \delta]\!]$ by $\mathcal{R}$ (*i.e.,* the dioid of the equivalence classes of $\mathcal{R}$). It is a complete dioid with the bottom element $\varepsilon = \gamma^{+\infty}\delta^{-\infty}$ and the top element $\top = \gamma^{-\infty}\delta^{+\infty}$.

The counter function canonically associated with a series s in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is the unique non-decreasing function $\mathcal{C}_s : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, +\infty\}$ such that $s = \bigoplus_{t\in\mathbb{Z}} \gamma^{\mathcal{C}_s(t)}\delta^t$ (with the convention $\gamma^{+\infty} = \varepsilon$ and $\gamma^{-\infty} = (\gamma^{-1})^*$). The $(\min, +)$-algebra is canonically associated with counters in TEG. Then, the order relation for counters and the classical order relation are dual. The valuation in γ of s, denoted $val_\gamma(s)$, is defined as the lower bound of $\{\mathcal{C}_s(t) > -\infty | t \in \mathbb{Z}\}$.

When an element s of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is used to code information concerning a transition of a TEG, then a monomial $\gamma^k\delta^t$ with $k, t \geqslant 0$ may be interpreted as *at most k events occur up to time t* and the associated counter function $\mathcal{C}_s$ as *at most* $\mathcal{C}_s(t)$ *events occur up to time t*. A polynomial is defined as a finite sum of monomials of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

In $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, a TEG can be described by the following model:

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx \end{cases} \tag{4}$$

where $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^n$ is the internal state, $u \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^p$ the input, and $y \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^m$ the output. Then, $A \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times n}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times p}$, and $C \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{m\times n}$.

The transfer function matrix H of the system is calculated with Th. 6:

$$H = CA^*B \tag{5}$$

In the following, interesting classes of series of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ are introduced.

Definition 16. (Periodicity). A series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is said to be periodic if it can be written as $s = p \oplus qr^*$ with p and q two polynomials and r a monomial. A matrix is said to be periodic if all its entries are periodic.

Definition 17. (Causality). A series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is said to be causal either if $(s = \varepsilon)$ or $(val_\gamma(s) \geqslant 0$ and $s \succeq \gamma^{val_\gamma(s)})$. The set of causal elements of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is a complete subdioid of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ denoted $\mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$. A matrix is said to be causal if each of its entries is causal.

Proposition 18. (Cottenceau et al. (1999)). The injection $i : \mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!] \to \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, $x \mapsto x$ is residuated and its residual is a projector denoted Pr_+. The causal projection of a matrix $M \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times m}$ is composed of the causal projections of its entries.

Remark 19. By definition of a residuated mapping, $\mathsf{Pr}_+(y)$ is the least upper bound of $\left\{x \in \mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!] | i(x) = x \preceq y\right\}$ and belongs to this subset. Then, $\mathsf{Pr}_+(y)$ is the greatest element of $\mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$ less or equal to $y \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

Definition 20. (Realizability). A matrix H with entries in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is said to be realizable if there exist four matrices $A1$, $A2$, B and C with entries in $\{\varepsilon, e\}$ such that $H = C(\gamma A1 \oplus \delta A2)^* B$.

In other words, H is realizable if there exists a TEG, the transfer function matrix of which is H. The following theorem recalls that the input-output relation of a TEG is characterized by periodicity and causality.

Theorem 21. (Cohen et al. (1989)). Let H be a matrix with entries in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The following statements are equivalent:

- H is periodic and causal.
- H is realizable.

Example 22. A manufacturing system,composed of three machines M_1 (with transitions x_1 and x_2), M_2 (with transitions x_3 and x_4) and M_3 (with transitions x_5 and x_6), is considered. M_1 and M_2 produce parts, which are pairwise assembled in M_3. The system is modelled by the TEG represented in Fig. 1.

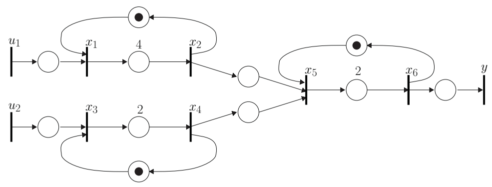

Fig. 1. Manufacturing system

In the following, the construction of the state-space model (4) is explained. For the state matrix A, A_{ij} is obtained by considering the direct link from x_j to x_i. For example, there are no direct links from x_1 to x_3, then $A_{31} = \varepsilon$. From x_1 to x_2, there is a direct link with 0 tokens and a holding time of 4 time units , then the k-th firing of x_2 happens at the earliest 4 time units after the k-th firing of x_1. This relation is coded in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ by $\gamma^0\delta^4 = \delta^4$ (*i.e.*, $A_{21} = \delta^4$). Dually, from x_2 to x_1, there is a direct link with one token and a holding time of 0 time units, then the $(k+1)$-st firing of x_1 happens at the earliest 0 time units after the k-th firing of x_2. This relation is coded in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ by $\gamma\delta^0 = \gamma$ (*i.e.*, $A_{12} = \gamma$). The following state matrix is obtained:

$$A = \begin{pmatrix} \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \delta^4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & e & \varepsilon & \gamma \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon \end{pmatrix} \tag{6}$$

Similar construction rules lead to the following input and output matrices:

$$B = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}^\intercal \tag{7}$$

$$C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix} \tag{8}$$

The transfer function matrix is:

$$H = CA^*B = \left(\delta^6 \left(\gamma\delta^4\right)^* \;\; \delta^4 \left(\gamma\delta^2\right)^*\right) \tag{9}$$

4. SECOND ORDER THEORY

In this section, stock is formally introduced and its constant tightest bounds are calculated. The results presented in this part come mainly from (MaxPlus (1991)).

Definition 23. (Stock Variation, MaxPlus (1991)). Let s_1 (resp. s_2) be a periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ representing the transfer relation from transition u to transition x_1 (resp. x_2). The stock variation from transition x_1 to transition x_2, *i.e.,* the difference between the number of tokens in any path from transition x_1 to transition x_2 at time t and the same quantity at time 0, $S_{s_1 s_2}$, is defined as:

$$S_{s_1s_2}(t) = \mathcal{C}_{s_1}(t) \phi \mathcal{C}_{s_2}(t) \tag{10}$$
$$= \mathcal{C}_{s_1}(t) - \mathcal{C}_{s_2}(t) \tag{11}$$

where $\mathcal{C}_{s_i}$ is the counter function associated with s_i.

Definition 24. (Stock Variation Matrix). Let u (resp. v) be an n-dimensional (resp. p-dimensional) vector with entries in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The associated vector of counter functions is $\mathcal{C}_u$ (resp. $\mathcal{C}_v$). The stock matrix $S_{uv}(t)$ at time t from u to v is defined by $S_{uv}(t) = \mathcal{C}_u(t) \phi \mathcal{C}_v(t)$.

Remark 25. For a TEG, the stock matrix is defined as $S_{xx}(t)$. If there is a path from transition x_i to transition x_j, $(S_{xx}(t))_{ij}$ represents the variation, from time 0 to time t, of the number of tokens in the considered path. This quantity does not depend on the choosen path.

The following theorem gives the constant (with respect to t) tightest upper and lower bounds for the stock matrix. It is known as stock evaluation formula.

Theorem 26. (MaxPlus (1991)). Let u (resp. v) be an n-dimensional (resp. p-dimensional) vector with entries in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. For all $t \in \mathbb{Z}$:

$$-\mathcal{C}^{\mathsf{T}}_{v \phi u}(0) \leqslant S_{uv}(t) \leqslant \mathcal{C}_{u \phi v}(0) \tag{12}$$

Moreover $-\mathcal{C}^{\mathsf{T}}_{v \phi u}(0)$ and $\mathcal{C}_{u \phi v}(0)$ are the tightest constant bounds.

Remark 27. According to the stock evaluation formula, the tightest constant (with respect to t) bounds for $S_{xx}(t)$ are:

$$-\mathcal{C}^{\mathsf{T}}_{x \phi x}(0) \leqslant S_{xx}(t) \leqslant \mathcal{C}_{x \phi x}(0) \tag{13}$$

In the following, $\mathcal{D}$ is assumed to be a complete dioid. The next theorem, known as increasing correlation principle, leads to bounds for the stock matrix, which are also constant with respect to the input and the perturbation.

Theorem 28. (MaxPlus (1991)). Let $H \in \mathcal{D}^{m\times p}$, and $u, v \in \mathcal{D}^p$. Then:

$$(Hu) \phi (Hv) \succeq (v \backslash u)(H \phi H) \tag{14}$$

Corollary 29. Let $H \in \mathcal{D}^{m\times p}$. Then, for all $u \in \mathcal{D}^p$:

$$(Hu) \phi (Hu) \succeq H \phi H \tag{15}$$

Proposition 30. Let $H \in \mathcal{D}^{m\times p}$ and $u \in \mathcal{D}^p$. The tightest constant (with respect to u) lower bound of $(Hu) \phi (Hu)$ is $H \phi H$.

Proof. According to Cor. 29:

$$(Hu) \phi (Hu) \succeq H \phi H \quad \forall u \in \mathcal{D}^p \tag{16}$$

Then, considering unit vectors leads directly to the results.

The state equation for TEG can be modified to take into account perturbations, by adding a vector $q \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^n$ to the state:

$$x = Ax \oplus Bu \oplus q \tag{17}$$
$$= A^*Bu \oplus A^*q \text{ see Th. 6} \tag{18}$$
$$= A^*\bar{B}\bar{u} \tag{19}$$

where $\bar{B} = (B \quad \mathsf{Id}_n)$ and $\bar{u} = (u^{\mathsf{T}} \quad q^{\mathsf{T}})^{\mathsf{T}}$.

Then, the tightest constant bounds of $S_{xx}(t)$ with respect to t, u and q are given by:

$$-\mathcal{C}^{\mathsf{T}}_{(A^*\bar{B}) \phi (A^*\bar{B})}(0) \leqslant S_{xx}(t) \leqslant \mathcal{C}_{(A^*\bar{B}) \phi (A^*\bar{B})}(0) \tag{20}$$

Remark 31. $\mathcal{C}_{(A^*\bar{B}) \phi (A^*\bar{B})}(0) \geqslant 0$, as $S_{xx}(t) = 0$ for all t if $u = \varepsilon$ and $q = \delta^{+\infty}$.

5. FEEDBACK CONTROLLER

5.1 Synthesis

For feedback synthesis, the influence of the perturbation is neglected ($q = \varepsilon$). Methods to take perturbations into account via disturbance decoupling exist (Lhommeau et al. (2002)), however this leads to a lower feedback controller and to a greater stock.

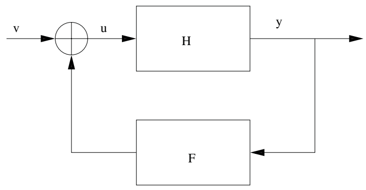

Fig. 2. TEG with an output feedback controller F

In the following, a linear output feedback controller is considered: $u = Fy \oplus v$ with v an external input. Inserting this control law into the model (4) provides (see Fig. 2):

$$y = HFy \oplus Hv \tag{21}$$

According to Th. 6, the least solution of (21) is $y = (HF)^* Hv$ where $H = CA^*B$ is the open-loop transfer function matrix . Then, the closed-loop transfer function matrix is $(HF)^* H$.

Theorem 32. (Cottenceau et al. (1999)). The greatest linear and causal output feedback controller such that $(HF)^* H = H$ is given by:

$$F_c = \mathsf{Pr}_+ (H \backslash H \phi H) \tag{22}$$

The previous feedback controller will delay the input as much as possible while preserving the same output as the uncontrolled system, whatever be the input. Below, a feedback controller taking into account a specific reference input w is proposed.

Proposition 33. For a given input w, the greatest linear and causal output feedback controller such that $(HF)^* Hw = Hw$ is given by:

$$F_w = \mathsf{Pr}_+ (H \backslash (Hw) \phi (Hw)) \tag{23}$$

Proof. As $\forall F, (HF)^* \succeq \mathsf{Id}$, $(HF)^* Hw \succeq Hw$. Then, the goal is to find the greatest causal output feedback controller F, such that:

$$(HF)^* Hw \preceq Hw \tag{24}$$

Thus:

$$(HF)^* Hw \preceq Hw \tag{25}$$
$$\Leftrightarrow (HF)^* \preceq (Hw) \phi (Hw) \tag{26}$$
$$\Leftrightarrow HF \preceq (Hw) \phi (Hw) \text{ Th. 14 and Prop. 15} \tag{27}$$
$$\Leftrightarrow F \preceq H \backslash (Hw) \phi (Hw) \tag{28}$$

Besides, according to Rem. 19:

$$F_w \preceq H \backslash (Hw) \phi (Hw) = \tilde{F}_w \tag{29}$$

Then:

$$Hw \preceq (HF_w)^* Hw \preceq \left(H\tilde{F}_w\right)^* Hw \preceq Hw \tag{30}$$

Consequently, $(HF_w)^* Hw = Hw$ and it is clear from Rem. 19 that F_w is the greatest causal element achieving equality.

Remark 34. An interesting particular case is to consider the specific input $w = e$, which corresponds to an impulse, as $e = \gamma^0\delta^0 = \gamma^0\delta^0 \oplus \gamma^1\delta^0 \oplus \ldots$ *i.e.,* an infinity of tokens is put in the system at $t = 0$. This input leads to the fastest output and yields the following feedback controller:

$$F_e = \mathsf{Pr}_+ \left(H \circ\!\!\!\!\backslash (He) \circ\!\!\!\!/ (He)\right) \tag{31}$$

Remark 35. Consider a feedback, which uses linear combinations (in the sense of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$) of the output:

$$u = F\hat{C}y \oplus v \tag{32}$$

It is possible to calculate the greatest linear and causal feedback $\hat{F}_w$, which preserves the response to w. Using a proof similar to the one of Prop. 33 leads to:

$$\hat{F}_w = \mathsf{Pr}_+ \left(H \circ\!\!\!\!\backslash (Hw) \circ\!\!\!\!/ \left(\hat{C}Hw\right)\right) \tag{33}$$

The following proposition gives a comparison between F_w and F_c.

Proposition 36. The feedback controller F_w, taking into account the reference input w, is greater or equal to the one recalled in Th. 32.

Proof. According to Cor. 29:

$$(Hw) \circ\!\!\!\!/ (Hw) \succeq H \circ\!\!\!\!/ H \tag{34}$$

Then, by dividing on the left by H and applying the causal projection (both are isotone mappings):

$$F_w \succeq F_c \tag{35}$$

Remark 37. Considering the reference input w:

$$(HF_w)^* Hw = Hw = (HF_c)^* Hw \tag{36}$$

Thus, feedback controllers F_w and F_c are equivalent for the reference input w.

An interesting problem is to extend the previous remark by finding *a set* of inputs which lead to the same output with feedback controllers F_w and F_c. The following proposition gives a first answer to this problem.

Proposition 38. Considering a specific reference input w, feedback controllers F_c and F_w lead to the same closed-loop response Hw for all inputs w' such that $w \preceq w' \preceq w_{\max}$ with:

$$w_{\max} = \left((HF_w)^* H\right) \circ\!\!\!\!\backslash (Hw) \tag{37}$$

Proof. First, as $(HF_w)^* Hw = Hw$, $w \preceq w_{\max}$. Then, the considered set of inputs is not empty. The second step consists in showing that:

$$Hw = (HF_c)^* Hw' = (HF_w)^* Hw' \tag{38}$$

Obviously, as $w \preceq w'$ and $F_c \preceq F_w$:

$$Hw \preceq (HF_c)^* Hw' \preceq (HF_w)^* Hw' \tag{39}$$

Besides, as $w' \preceq w_{\max}$:

$$(HF_w)^* Hw' \preceq (HF_w)^* Hw_{\max} \tag{40}$$

$$\preceq Hw \tag{41}$$

5.2 Performance Analysis

In the following, a formal relation is presented between the output feedback controller and the tightest bounds for the stock matrix.

Proposition 39. Let $A_1, A_2 \in \mathcal{D}^{n\times n}$ and $\bar{B} = (B \quad \mathsf{Id}_n)$ with $B \in \mathcal{D}^{n\times m}$. If $A_1 \succeq A_2$, then $\left(A_1^*\bar{B}\right) \circ\!\!\!\!/ \left(A_1^*\bar{B}\right) \succeq \left(A_2^*\bar{B}\right) \circ\!\!\!\!/ \left(A_2^*\bar{B}\right)$.

Proof. As $A^*\bar{B} = (A^*B \quad A^*)$, according to Prop. 13:

$$\left(A^*\bar{B}\right) \circ\!\!\!\!/ \left(A^*\bar{B}\right) = (A^*B) \circ\!\!\!\!/ (A^*B) \wedge A^* \circ\!\!\!\!/ A^* \tag{42}$$

Clearly $A^* \circ\!\!\!\!/ A^* = A^*$, as $XA^* \preceq A^*$ implies $X \preceq A^*$ and $A^*A^* = A^*$. Besides, $(A^*B) \circ\!\!\!\!/ (A^*B) \succeq A^*$, as $A^*A^*B = A^*B$. Consequently:

$$\left(A^*\bar{B}\right) \circ\!\!\!\!/ \left(A^*\bar{B}\right) = A^* \tag{43}$$

Then, $A_1 \succeq A_2$ implies:

$$\left(A_1^*\bar{B}\right) \circ\!\!\!\!/ \left(A_1^*\bar{B}\right) \succeq \left(A_2^*\bar{B}\right) \circ\!\!\!\!/ \left(A_2^*\bar{B}\right) \tag{44}$$

Proposition 40. Given a TEG and a reference input w, the tightest bounds for the stock matrix with feedback controller F_w are at least as strict as with the classical feedback controller F_c.

Proof. For a TEG with an output feedback controller:

$$x = Ax \oplus Bu \oplus q \tag{45}$$

$$= (A \oplus BFC)\, x \oplus Bv \oplus q \tag{46}$$

Then, the closed-loop state matrix is $A \oplus BFC$. As $F_w \succeq F_c$ (Prop. 36), the closed-loop state matrix A_w with feedback controller F_w is greater or equal to A_c, the one with F_c. Then, Prop. 39 can be direcly applied:

$$0 \leqslant \mathcal{C}_{\left(A_w^*\bar{B}\right) \circ\!\!\!\!/ \left(A_w^*\bar{B}\right)}(0) \leqslant \mathcal{C}_{\left(A_c^*\bar{B}\right) \circ\!\!\!\!/ \left(A_c^*\bar{B}\right)}(0) \tag{47}$$

$$0 \geqslant -\mathcal{C}^{\intercal}_{\left(A_w^*\bar{B}\right) \circ\!\!\!\!/ \left(A_w^*\bar{B}\right)}(0) \geqslant -\mathcal{C}^{\intercal}_{\left(A_c^*\bar{B}\right) \circ\!\!\!\!/ \left(A_c^*\bar{B}\right)}(0) \tag{48}$$

The signs of the upper and lower bounds are obtained considering Rem. 31.

5.3 Application

For the TEG of Ex. 22, the impulse input is considered, *i.e.*, $w = e$.

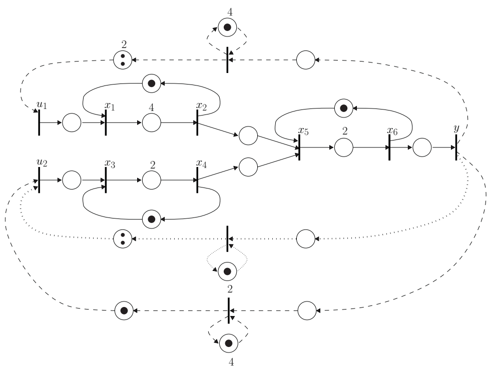

Fig. 3. Manufacturing system with feedback controller F_c (resp. F_w) drawn with dotted lines (resp. dashed lines)

The calculations below have been done with the software described in(Cottenceau et al. (2000)) and (Hardouin et al. (2001)). The following output feedback controllers are obtained:

$$F_c = \begin{pmatrix} \varepsilon \\ \gamma^2 \left(\gamma\delta^2\right)^* \end{pmatrix} \tag{49}$$

$$F_e = \begin{pmatrix} \gamma^2\delta^2 \left(\gamma\delta^4\right)^* \\ \gamma \left(\gamma\delta^4\right)^* \end{pmatrix} \tag{50}$$

As expected, $F_e \succeq F_c$. A realization of F_c (resp. F_e) is drawn in Fig. 3 with dotted lines (resp. dashed lines).

The closed-loop transfer function matrices are:

$$(HF_c)^* H = H \tag{51}$$
$$= \left(\delta^6 \left(\gamma\delta^4\right)^* \ \delta^4 \left(\gamma\delta^2\right)^* \right) \tag{52}$$
$$(HF_e)^* H = \left(\delta^6 \left(\gamma\delta^4\right)^* \ \delta^4 \left(\gamma\delta^4\right)^* \right) \tag{53}$$

As expected, the system controlled by F_e is slower than the one controlled by F_c. But both closed-loop systems have the same response for all inputs e' such that $e \preceq e' \preceq e_{\max}$ with:

$$e_{\max} = \left((HF_e)^* H\right) \circ\backslash (He) \tag{54}$$
$$= \begin{pmatrix} \left(\gamma\delta^4\right)^* \\ \delta^2 \left(\gamma\delta^4\right)^* \end{pmatrix} \tag{55}$$

For all e' such that $e \preceq e' \preceq e_{\max}$ the following response is obtained:

$$He = \delta^6 \left(\gamma\delta^4\right)^* \tag{56}$$

The response of the system controlled by F_e can be strictly greater than the one of the system controlled by F_e, as shown with the following input:

$$v = \begin{pmatrix} e \oplus \gamma^3\delta^{+\infty} \\ \delta^6 \oplus \gamma^3\delta^{+\infty} \end{pmatrix} \tag{57}$$

Then, the outputs are:

$$(HF_c)^* Hv = \delta^{10} \oplus \gamma\delta^{12} \oplus \gamma^2\delta^{14} \oplus \gamma^3\delta^{+\infty} \tag{58}$$
$$(HF_e)^* Hv = \delta^{10} \oplus \gamma\delta^{14} \oplus \gamma^2\delta^{18} \oplus \gamma^3\delta^{+\infty} \tag{59}$$
$$\tag{60}$$

Upper and lower bounds are calculated for $S^c_{xx}(t)$ (resp. $S^e_{xx}(t)$) considering feedback controller F_c (resp. F_e):

$$\begin{pmatrix} 0 & 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & -2 & -2 & 0 & 0 \\ -\infty & -\infty & 0 & 0 & 0 & 0 \\ -\infty & -\infty & -1 & 0 & 0 & 0 \\ -\infty & -\infty & -2 & -2 & 0 & 0 \\ -\infty & -\infty & -2 & -2 & -1 & 0 \end{pmatrix} \leqslant S^c_{xx}(t) \leqslant \begin{pmatrix} 0 & 1 & +\infty & +\infty & +\infty & +\infty \\ 0 & 0 & +\infty & +\infty & +\infty & +\infty \\ 2 & 2 & 0 & 1 & 2 & 2 \\ 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{61}$$

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 & 0 \\ -2 & -2 & -1 & 0 & 0 & 0 \\ -2 & -2 & -1 & -1 & 0 & 0 \\ -2 & -2 & -1 & -1 & -1 & 0 \end{pmatrix} \leqslant S^e_{xx}(t) \leqslant \begin{pmatrix} 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{62}$$

The bounds obtained for $S^e_{xx}(t)$ are stricter than the ones obtained for $S^c_{xx}(t)$. Thus, the highest stock for the system controlled by F_e is lower than the highest stock for the system controlled by F_c. For example, a failure on machine M_3 (*i.e.,* a perturbation on transition x_5 or transition x_6) could cause an unexpected accumulation of tokens in the place from transition x_2 to transition x_5. The size of this accumulation is bounded by:

$$\left(\mathcal{C}_{(A_{cl}^*\bar{B})\not\circ(A_{cl}^*\bar{B})}(0)\right)_{25} \tag{63}$$

where A_{cl} is the closed-loop state matrix. It is equal to $A_e = A \oplus BF_eC$ if feedback controller F_e is considered or to $A_c = A \oplus BF_cC$ if feedback controller F_c is considered. According to (61) and (62), the accumulation of tokens in the place from transition x_2 to transition x_5 is not bounded considering F_c, while it is limited to two tokens considering F_e.

Besides, in this example, F_e ensures also internal stability (stock remains bounded), as all entries of $\mathcal{C}_{(A_e^*\bar{B})\not\circ(A_e^*\bar{B})}(0)$ are finite.

6. CONCLUSION

First, the greatest causal output feedback controller F_w preserving the response of the system for a specific, predefined reference input w is introduced. It turns out that this controller preserves the response not only for w, but for a whole set of inputs. Second, it is shown that the bounds for the stock are at least as strict with F_w as with the standard greatest causal output feedback controller, which preserves the response of the system whatever be the input.

REFERENCES

Baccelli, F., Cohen, G., Olsder, G.J., and Quadrat, J.P. (1992). *Synchronization and Linearity, An Algebra for Discrete Event Systems*. John Wiley and Sons, New York, USA.

Blyth, T.S. and Janowitz, M.F. (1972). *Residuation Theory*. Pergamon press, Oxford, United Kingdom.

Cohen, G., Moller, P., Quadrat, J.P., and Viot, M. (1989). Algebraic tools for the performance evaluation of discrete event systems. *Proceedings of the IEEE*, 77(1), 39–58. Special issue on Discrete Event Systems.

Cottenceau, B., Hardouin, L., Boimond, J.L., and Ferrier, J.L. (1999). Synthesis of greatest linear feedback for timed event graphs in dioid. *IEEE Transactions on Automatic Control*, 44(6), 1258–1262. Available at `doi:10.1109/9.769386`.

Cottenceau, B., Hardouin, L., Boimond, J.L., and Ferrier, J.L. (2001). Model reference control for timed event graphs in dioids. *Automatica*, 37(9), 1451–1458. Available at `doi:10.1016/S0005-1098(01)00073-5`.

Cottenceau, B., Hardouin, L., Lhommeau, M., and Boimond, J.L. (2000). Data processing tool for calculation in dioid. In *Proceedings of the 5th International Workshop on Discrete Event Systems, WODES 2000*. Ghent, Belgium. Available at `istia.univ-angers.fr/ hardouin/outils.html`.

Hardouin, L., Cottenceau, B., and Lhommeau, M. (2001). Software tools for manipulating periodic series. Available at `istia.univ-angers.fr/ hardouin/outils.html`.

Lhommeau, M., Hardouin, L., and Cottenceau, B. (2002). Disturbance decoupling of timed event graphs by output feedback controller. In *Proceedings of the 6th International Workshop on Discrete Event Systems, WODES'02*. Zaragoza, Spain. Available at `istia.univ-angers.fr/ hardouin/Wodes02.pdf`.

MaxPlus (1991). Second order theory of min-linear systems and its application to discrete event systems. In *Proceedings of the 30th IEEE Conference on Decision and Control*, volume 2, 1511 – 1516. Brighton, United Kingdom.