Exercise 3 — Solution

Exercise 3.1

Inequalities for the specifications:

\[ x_2(k) \leq 1 , \]
\[ x_5(k) \leq 1 , \]
\[ x_1(k) + x_4(k) \leq 1 , \]

or, in matrix form,

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x(k) \\
\Gamma \\
\end{bmatrix}
\leq
\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}.
\]

Initial marking of the controller places:

\[ x_c^0 = b - \Gamma x^0 = [1 \ 1 \ 1]' . \]

Connections between controller places and plant transitions (let \( A \) be the incidence matrix of the plant model):

\[ A_c = -\Gamma A =
\begin{bmatrix}
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
-1 & 1 & 0 & -1 & 1 & 0 & 0 \\
\end{bmatrix} . \]

The Petri net comprising both plant and controller is shown in Figure 1.

![Figure 1: Petri net with plant and controller (Exercise 4.3).](image-url)
Exercise 3.2

a) What needs to be verified is whether the specified inequalities can be violated under the action of the given controller; in this case, adding the controller to the Petri net model of the plant, it can easily be seen that the inequalities cannot be violated, so the controller does enforce the specification. One can also check this fact more formally by constructing the reachability graph for the complete model (plant + controller) and inspecting if the inequalities are breached in any of the nodes; in this case, they are not.

However, it is also easy to see that the given controller is overly restrictive, more specifically with respect to the third inequality. Whereas the specification only requires that the number of tokens in places 1, 2, 3, and 4 never exceeds 4, the controller enforces that the maximum number of firings of transitions $t_1$ and $t_3$ combined is 4, which means that once these two transitions together fire four times, they will never be able to fire again. This fact can also be checked more formally, for instance by computing the ideal (minimally restrictive) controller for the specification. If the given controller is not identical to the ideal one, since the latter is unique one can conclude that the given controller is not minimally restrictive.

b) The question is what is the maximum number of times transition $t_6$ can fire. As each firing of $t_6$ requires one token in $p_5$, which can only arrive due to the firing of $t_5$, one can equivalently investigate how many times $t_5$ can fire. Each firing of $t_5$, in turn, requires one token in $p_2$ and one in $p_4$, which ultimately depend on the firing of $t_1$ and $t_3$, respectively. So, the best that can be done is to fire each of $t_1$ and $t_3$ twice, and then fire $t_5$ and $t_6$ also twice. The system can, therefore, produce at most 2 products under the action of the given controller.

c) A possible modified controller incidence matrix is

$$
\tilde{A}_c = \begin{bmatrix}
0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 2 & 0
\end{bmatrix}
$$

Since this controller is identical to the ideal one (computed using the standard procedure), it is in fact minimally restrictive.

Exercise 3.3

a) Inequalities for the specifications:

$$
5x_1(k) + 2x_2(k) \leq 20,
$$

$$
x_4(k) \leq 1,
$$

$$
-x_1(k) + x_2(k) \leq 0,
$$

or, in matrix form,

$$
\begin{bmatrix}
5 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 0
\end{bmatrix} \Gamma x(k) \leq \begin{bmatrix} 20 \\ 1 \\ 0 \end{bmatrix}.
$$

Initial marking of the controller places: $x_0^0 = b - \Gamma x^0 = [15 \ 1 \ 1]'$.

Controller matrix:

$$
A_c = -\Gamma A = \begin{bmatrix}
-5 & -2 & 7 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 \\
1 & -1 & 0 & 0 & 0
\end{bmatrix}.
$$

b) The specifications are not ideally enforceable: $-\Gamma A_{\text{douc}} = [0 \ -1 \ 0]' \neq 0$, therefore condition (2.24) is violated (see Lecture Notes, page 33).