Proceedings of the 13th International Workshop on Discrete Event Systems, Xi'an, China, May 30 - June 1, 2016

Modeling and Control of Resource Sharing Problems in Dioids

Soraia Moradi[1,2,3], Laurent Hardouin[3], and Jörg Raisch[1,2]

***Abstract*— The topic of this paper is the modeling and control of a class of timed Petri nets with resource sharing problems in a dioid framework. We first introduce a signal which denotes the number of resources available for each competing subsystem at each instant of time. Based on this signal, the overall system is modeled in min-plus algebra. Using residuation theory, an optimal control policy is developed, where optimality is in the sense of a lexicographical order reflecting the chosen prioritization of subsystems.**

I. INTRODUCTION

Timed event graphs (TEG)s are a subclass of timed Petri nets where each place has exactly one upstream and one downstream transition and all arcs have weight 1. The time/event behavior of TEGs, under the earliest functioning rule (i.e., transitions fire as soon as they are enabled), can be expressed linearly over some dioids [1]. TEGs can only model synchronization but not concurrency or choice. In many applications, like railway networks and manufacturing systems, there are only limited resources, which are shared among different users. For example, in a railway network, there may be single track segments which are used by multiple trains, but, at each instant of time, only one train can occupy the track. This problem is called the "Resource Sharing" (RS) problem. Systems with RS problems can be modeled by timed Petri nets but not by TEGs, as they contain choice or conflict. In the literature, various methods have been investigated to deal with the RS problem. In [2], systems with RS are modeled by switching max-plus linear systems, where a system can switch between different modes of operation and in each mode is modeled by a linear max-plus system. Using model predictive control (MPC), the optimal switching sequence is obtained. In [3], modeling and control of switching max-plus-linear systems with random and deterministic switching have been discussed. In [4], the just in time control problem of switching max-plus linear systems where the switching variable on the study horizon is given is considered. In [5], the model consists of a TEG and some additional inequalities which model the limited availability of shared resources. In [6], conflicting time event graphs are modeled in the max-plus algebra and an approach to calculate the cycle time is proposed. In [7], modeling and performance evaluation of timed Petri nets with different levels of priority are investigated. Three possible place/transition patterns are considered, namely, conflict, synchronization and priority configurations. In [8], systems with RS problems are modeled in the max-plus algebra. A method to detect conflicts by checking the time line overlaps of processes is introduced. In order to solve conflicts, the schedule is changed to move up the process with low priority.

In this paper, a method to model and control the RS problem in the min-plus algebra is proposed. The main contribution of this work is as follows. First, a signal denoting the number of resources available for competing subsystems, at each instant of time is introduced. The definition of this signal incorporates a predefined prioritization policy. Based on this signal, the overall system is modeled in min-plus algebra. Using residuation theory, an optimal control policy is developed, where optimality is in the sense of a lexicographical order reflecting the chosen prioritization of subsystems. In essence, we are aiming at firing input transitions of subsystems as late as possible, while making sure that the firing of output transitions is not later than specified in given reference signals. Moreover, the control of lower-priority subsystems may not degrade the performance of higher-priority subsystems.

The paper is organized as follows. Section II recalls the necessary algebraic tools. In Section III, modeling of a system with RS problem over the dioid $\overline{\mathbb{Z}}_{min}$ is discussed. Section IV addresses the optimal control problem, and Section V provides some conclusions.

II. ALGEBRAIC PRELIMINARIES

The following is a summary of basic results from dioid theory and residuation theory. The interested reader is invited to peruse [1], [9], and [10] for more details.

A. Dioid Theory

A dioid $\mathcal{D}$ is a set endowed with two internal operations denoted $\oplus$ (addition) and $\otimes$ (multiplication), both associative and having a neutral element denoted ε (zero element) and e (unit element), respectively. Moreover, $\oplus$ is commutative and idempotent ($\forall a \in \mathcal{D}, a \oplus a = a$), $\otimes$ distributes over $\oplus$, and ε is absorbing for $\otimes$ ($\forall a \in \mathcal{D}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). By convention, multiplication is often expressed by juxtaposition, i.e., $a \otimes b = ab$. The operation $\oplus$ induces an order relation $\preceq$ on $\mathcal{D}$, defined by: $\forall a, b \in \mathcal{D}, a \preceq b \Leftrightarrow a \oplus b = b$. A diolid is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. In this case, the greatest (in the sense of the above order) element

[1] Fachgebiet Regelungssysteme, Technische Universität Berlin, Einsteinufer 17, 10587 Berlin, Germany,
[2] Control Systems Group, Max Planck Institute for Dynamics of Complex Technical Systems , Magdeburg, Germany,
[3] Laboratoire d'Ingénierie des Systèmes Automatisés, Université d'Angers. 62, Avenue Notre Dame du Lac, 49000 Angers, France. moradi@control.tu-berlin.de, laurent.hardouin@univ-angers.fr, raisch@control.tu-berlin.de

978-1-5090-4190-9/16/$31.00 ©2016 IEEE

of $\mathcal{D}$ is denoted $\top$ (the top element) and is equal to the sum of all its elements ($\top = \bigoplus_{x\in\mathcal{D}} x$). In a complete dioid, another binary operation ("greatest lower bound") denoted $\wedge$, can be defined by $a \wedge b = \bigoplus_{x\in\mathcal{D}_{ab}} x$ with $\mathcal{D}_{ab} = \{x \in \mathcal{D} | x \preceq a \quad \text{and} \quad x \preceq b\}$.

- The set $\overline{\mathbb{Z}}_{min} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the standard $\min$ operator as $\oplus$ and standard addition as $\otimes$ is a complete dioid, where $\varepsilon = +\infty$, $e = 0$ and $\top = -\infty$. Consequently, $\preceq$ on $\overline{\mathbb{Z}}_{min}$ is the reverse of the standard order (e.g., $3 \succeq 5$), and the greatest lower bound $\wedge$ is the standard $\max$ operator. Both operations can be readily generalized for matrices of appropriate dimensions:

$$\forall A, B \in \overline{\mathbb{Z}}_{min}^{n\times m}, (A \oplus B)_{ij} = A_{ij} \oplus B_{ij},$$

$$\forall A \in \overline{\mathbb{Z}}_{min}^{n\times m}, B \in \overline{\mathbb{Z}}_{min}^{m\times p}, (A \otimes B)_{ij} = \bigoplus_{k=1}^{m} A_{ik} \otimes B_{kj}.$$

- Consider the set of formal power series in δ with exponents in $\mathbb{Z}$ and coefficients in $\overline{\mathbb{Z}}_{min}$, denoted $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$.
  - an element $s \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]$ can be interpreted as a map $s : \mathbb{Z} \to \overline{\mathbb{Z}}_{min}$ and written as:
  $$s = \bigoplus_{t\in\mathbb{Z}} s(t)\delta^t,$$
  - the set $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ can be equipped with operations $\oplus$ and $\otimes$ defined by: $\forall t \in \mathbb{Z}, s_1, s_2 \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]$:
  $$(s_1 \oplus s_2)(t) = s_1(t) \oplus s_2(t)$$
  $$(s_1 \otimes s_2)(t) = \bigoplus_{j\in\mathbb{Z}} s_1(j) \otimes s_2(t-j)$$
  - endowed with these operations, the set $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is a complete dioid with zero element $\varepsilon = \bigoplus_{t\in\mathbb{Z}} \varepsilon\delta^t$ and one element $e = \bigoplus_{t\in\mathbb{Z}} e(t)\delta^t$, $e(t) = \begin{cases} e, & t = 0 \\ \varepsilon & \text{otherwise.} \end{cases}$

  Note that $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ inherits its order from $\overline{\mathbb{Z}}_{min}$. From the general definition, $s_1 \preceq s_2 \Leftrightarrow s_1 \oplus s_2 = s_2$, it follows immediately that $s_1 \preceq s_2 \Leftrightarrow \forall t : s_1(t) \preceq s_2(t)$.
- $s \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is nonincreasing if
$$t_1 \le t_2 \Rightarrow s(t_2) \preceq s(t_1).$$
The set of nonincreasing formal power series in $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is denoted by $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ and is a complete dioid. Because of nonincreasingness, elements in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$, can be represented compactly . E.g., the series $s = s(t)\delta^t$ with
$$s(t) = \begin{cases} e, & t \le 0 \\ 1, & t = 1 \\ 2, & t = 2, 3, 4 \\ 3, & t \ge 5 \end{cases}$$
can be written as:
$$s = e\delta^0 \oplus 1\delta^1 \oplus 2\delta^4 \oplus 3\delta^{+\infty}.$$

Note that we use the same symbol to refer to multiplication in $\overline{\mathbb{Z}}_{min}$ and $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$. The same is true for addition, zero and one element.

Theorem 1: [10] Over a complete dioid $\mathcal{D}$, the implicit equation $x = ax \oplus b$ admits a least solution $x = a^*b$, where a^* is the Kleene star of a, defined by $a^* = \bigoplus_{i\in\mathbb{N}_0} a^i$ with $a^0 = e$.

B. Residuation Theory

Residuation theory (e.g., [11], [12]) provides, under some assumptions, the greatest solution (in accordance with the considered order) to the inequality $f(x) \preceq b$ where f is an order-preserving, or isotone, mapping (*i.e.*, $a \preceq b \Rightarrow f(a) \preceq f(b)$) defined over ordered sets.

Definition 1 (Residuation): Let $f : \mathcal{D} \to \mathcal{C}$ be an isotone mapping with $(\mathcal{D}, \preceq)$ and $(\mathcal{C}, \preceq)$ being ordered sets. Mapping f is said to be residuated if, for all $y \in \mathcal{C}$, the greatest element of the subset $\{x \in \mathcal{D} | f(x) \preceq y\}$ exists and lies in this subset. This element is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f.

Denote left multiplication by a in a dioid by L_a, *i.e.*, $L_a : x \mapsto a \otimes x$. Mapping L_a is residuated. Its residual is denoted $L_a^\sharp : x \mapsto a \backslash\!\!\circ x$ and called *left division by* a. Therefore, $a \backslash\!\!\circ b$ is the greatest solution to inequality $a \otimes x \preceq b$ (*i.e.* $a \backslash\!\!\circ b = \hat{x} = \bigoplus\{x \mid a \otimes x \preceq b\}$). Similarly, right multiplication by a is a residuated mapping. Its residual $x /\!\!\circ a$ is called right division by a.

Residuation can be extended to the matrix case. Given the matrices $A \in \mathcal{D}^{m\times n}$ and $B \in \mathcal{D}^{m\times p}$, the greatest solution of $A \otimes X \preceq B$, with $\preceq$ understood elementwise, is given by $D = A \backslash\!\!\circ B$, where

$$D_{ij} = \bigwedge_{k=1}^{m} (A_{ki} \backslash\!\!\circ B_{kj}).$$

III. Modeling

This section presents a modeling method for systems with RS. In a first step, RS is ignored and, the system is modeled linearly in $\overline{\mathbb{Z}}_{min}$. Then a signal α describing the availability of shared resources is introduced. Using α and taking into account a given priority policy, the model resulting from step 1 is modified to include RS effects.

A. Modeling of TEG

Timed event graphs can be seen as linear discrete event dynamical systems in suitable semirings (e.g., [10], [1]). For instance, by associating to each transition $\mathbf{x}_i$ a "counter" function $x_i : \mathbb{Z} \to \overline{\mathbb{Z}}_{min}$, where $x_i(t)$ is equal to the number of firings of transition $\mathbf{x}_i$ up to time t, it is possible to obtain a linear representation in $\overline{\mathbb{Z}}_{min}$.

A TEG can be modeled over $\overline{\mathbb{Z}}_{min}$ as:

$$\begin{cases} x(t) = A_1 x(t-1) \oplus ... \oplus A_T x(t-T) \oplus \\ \qquad B_0 u(t) \oplus ... \oplus B_M u(t-M) \\ y(t) = C \otimes x(t) \end{cases} \tag{1}$$

where $x(t) \in \overline{\mathbb{Z}}_{min}^n$, with n the number of internal transitions, $u(t) \in \overline{\mathbb{Z}}_{min}^p$ with p the number of input transitions and $y(t) \in \overline{\mathbb{Z}}_{min}^q$ with q the number of output transitions.

Matrices $A_1, ..., A_T$, $B_0, ..., B_M$ and C are of appropriate size with entries in $\overline{\mathbb{Z}}_{min}$. T is the maximum holding time of places between internal transitions and M denotes the maximum holding time of places, connecting input transitions to internal ones.
The counter functions x_i, u_m, y_j can be represented by nonincreasing formal power series, often referred to as δ-transforms, and the TEG model can be expressed in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ as:

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx, \end{cases} \tag{2}$$

where $x \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^n$, $u \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^p$ and $y \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^q$. Matrices A, B and C are of appropriate size with entries in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$.
According to Theorem 1, the least solution of (2) is $y = CA^*Bu$, where CA^*B is refered to as the system transfer function matrix. The entries of CA^*B are periodic series in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ [1].
A periodic series can be written as $s = \bar{p} \oplus \bar{q}(\nu\delta^\tau)^*$ where $\bar{p} = \bigoplus_{i=0}^{n_p} p_i\delta^i$ is a polynomial representing a transient, $\bar{q} = \bigoplus_{i=0}^{n_q} q_i\delta^i$ is a polynomial representing a pattern that is repeated every τ time units and after ν firings of the corresponding transition. The asymptotic slope $\sigma_\infty(s)$ of a periodic series is defined as $\sigma_\infty(s) = \frac{\nu}{\tau}$ and, in a manufacturing context, can be viewed as the production rate of the system.

Example 1: Consider the TEG shown in Fig.1, where we use the convention that holding times of places are 0 unless specified otherwise. Counters u, x_1 , x_2, and y are related

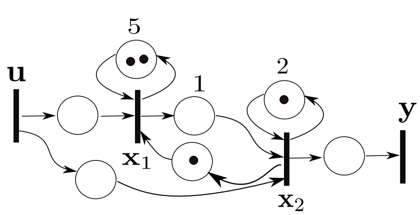

Fig. 1: A Single Input Single Output TEG [13].

as follows over $\overline{\mathbb{Z}}_{min}$:

$$\begin{cases} x_1(t) = 2 \otimes x_1(t-5) \oplus 1 \otimes x_2(t) \oplus u(t) \\ x_2(t) = x_1(t-1) \oplus 1 \otimes x_2(t-2) \oplus u(t) \\ y(t) = x_2(t), \end{cases}$$

Their respective δ-transforms are then related as:

$$\begin{cases} x_1 = 2\delta^5 x_1 \oplus 1x_2 \oplus u \\ x_2 = \delta x_1 \oplus 1\delta^2 x_2 \oplus u \\ y = x_2, \end{cases}$$

Consequently, by considering the state vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the following representation over $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ is obtained :

$$x = \begin{pmatrix} 2\delta^5 & 1 \\ \delta & 1\delta^2 \end{pmatrix} x \oplus \begin{pmatrix} e \\ e \end{pmatrix} u \tag{3}$$
$$y = \begin{pmatrix} \varepsilon & e \end{pmatrix} x.$$

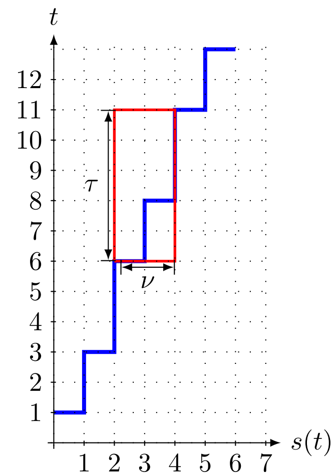

Fig. 2: Periodic series $(\delta^1 \oplus 1\delta^3)(2\delta^5)^*$

The transfer function matrix of this TEG is then given by:

$$H = (\delta \oplus 1\delta^3)(2\delta^5)^*, \tag{4}$$

which is graphically represented in Fig. 2. These computations can be done by the software introduced in [14] and [15].

B. Modeling of resource sharing problem

Consider a system in which different users (subsystems) share a limited number of (equal) resources. For notational simplicity, each subsystem is assumed to be a SISO TEG denoted S^k as shown in Fig. 3. In this system,

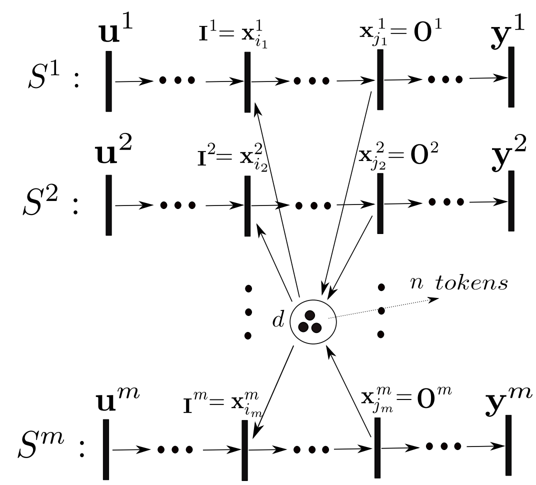

Fig. 3: n resources, shared between m users.

- m TEGs compete for n shared resources, modeled by n tokens in a shared place;
- the tokens corresponding to the shared resources have a holding time $d \geq 1$.
- allocation of a resource by subsystem S^k is modeled by the firing of transition $\mathbf{x}^k_{i_k}$, release of a resource by

the firing of $\mathbf{x}^k_{j_k}$. For simplification, these transitions are denoted by $\mathbf{I}^k$ and $\mathbf{O}^k$, respectively.

Neglecting resource sharing (which for each subsystem means ignoring all the arcs coming from other subsystems and ending in the shared place and ignoring all arcs beginning in the shared place and ending in other subsystems) and considering the earliest firing rule, the k-th subsystem can be modeled in $\overline{\mathbb{Z}}_{min}$ as:

$$\tilde{S}^k : \begin{cases} x^k(t) = A^k_1 x^k(t-1) \oplus ... \oplus A^k_{T_k} x^k(t-T_k) \\ \qquad \oplus B^k_0 u^k(t) \oplus ... \oplus B^k_{M_k} u^k(t-M_k) \\ y^k(t) = C^k \otimes x^k(t) \end{cases} \qquad (5)$$

In order to decide which user takes the resource in the case of conflict, we introduce a priority policy. In the case of conflict, the system with the higher priority will take the available resource.

Priority policy: In the following, we assume that the priority of $\tilde{S}^k$ is higher than $\tilde{S}^{k+1}$, for $k = 1, ..., m-1$. Then, the following recursive equation determines the number of resources available for $\tilde{S}^k$ at time t:

$$\begin{aligned} &\alpha^k(t) = \alpha^{k-1}(t) \otimes I^{k-1}(t-1) \oslash I^{k-1}(t) \quad \text{for} \quad k = 2, ..., m \\ &\alpha^1(t) = (\textstyle\bigotimes_{k=1}^m [O^k(t-d) \oslash I^k(t-1)]) \otimes n. \end{aligned} \qquad (6)$$

$\alpha^1(t)$ represents the number of resources available for $\tilde{S}^1$ at time instant t, which is equal to the initial number of resources, n, minus the number of resources which are already being used and therefore, not available. The quantity $O^k(t-d) \oslash I^k(t-1)$ represents the number of resources released by $\tilde{S}^k$ up to time $t-d$ minus the number of resources allocated by $\tilde{S}^k$ up to time $t-1$.
$\alpha^k(t), k = 2, ..., m$, represents the number of resources available for $\tilde{S}^k$ at time t. It is given by $\alpha^{k-1}(t)$ minus the number of resources newly allocated by $\tilde{S}^{k-1}$ at time t. The latter is given by, in standard algebra, $I^{k-1}(t) - I^{k-1}(t-1)$. Hence, in $\overline{\mathbb{Z}}_{min}$, (6) follows.
We next describe how the signal α^k affects the behavior of the k-th subsystem. $\tilde{S}^k$ under the influence of α^k is denoted by S^k, and it evolves according to:

$$S^k : \begin{cases} x^k(t) = (A^k_1 \oplus A^k_{\alpha(t)}) x^k(t-1) \oplus ... \oplus A^k_{T_k} x^k(t-T_k) \\ \qquad \oplus B^k_0 u^k(t) \oplus ... \oplus B^k_{M_k} u^k(t-M_k) \\ y^k(t) = C^k \otimes x^k(t) \end{cases} \qquad (7)$$

where

$$(A^k_\alpha(t))_{i,j} = \begin{cases} \alpha^k(t) & \text{if } i = j \text{ and } x^k_i = I^k \\ \varepsilon & \text{else} \end{cases} . \qquad (8)$$

Note that adding the term $A^k_{\alpha(t)}$ in (7) amounts, in conventional algebra, to the following statement: the number of firings of transition $\mathbf{x}^k_{i_k}$ in system S^k at time t, i.e., $x^k_{i_k}(t) - x^k_{i_k}(t-1)$ is additionally (when compared to the case without resource sharing) restrained by the term $\alpha^k(t)$.

Example 2: Consider the system shown in Fig. 4 under the priority policy discussed above.

Neglecting RS, each subsystem can be modeled in $\overline{\mathbb{Z}}_{min}$ as:

$$\begin{aligned} &\tilde{S}^1 : x^1(t) = A^1_2 \otimes x^1(t-2) \oplus A^1_3 \otimes x^1(t-3) \oplus B^1_0 u^1(t), \\ &\tilde{S}^2 : x^2(t) = A^2_2 \otimes x^2(t-2) \oplus A^2_5 \otimes x^2(t-5) \oplus B^2_0 u^2(t), \\ &\tilde{S}^3 : x^3(t) = A^3_2 \otimes x^3(t-2) \oplus A^3_6 \otimes x^3(t-6) \oplus B^3_0 u^3(t), \\ &\tilde{S}^4 : x^4(t) = A^4_2 \otimes x^4(t-2) \oplus A^4_4 \otimes x^4(t-4) \oplus B^4_0 u^4(t), \\ &\text{and} \quad y^k(t) = C^k \otimes x^k(t), \quad k = 1, ..., 4 \end{aligned} \qquad (9)$$

where $A^1_2 = A^2_2 = A^3_2 = A^4_2 = \begin{pmatrix} \varepsilon & 2 \\ \varepsilon & \varepsilon \end{pmatrix}$, $A^1_3 = A^2_5 = A^3_6 = A^4_4 = \begin{pmatrix} \varepsilon & \varepsilon \\ e & \varepsilon \end{pmatrix}$, $B^k_0 = \begin{pmatrix} e \\ \varepsilon \end{pmatrix}$ and $C^k = \begin{pmatrix} \varepsilon & e \end{pmatrix}, k = 1, ..., 4.$

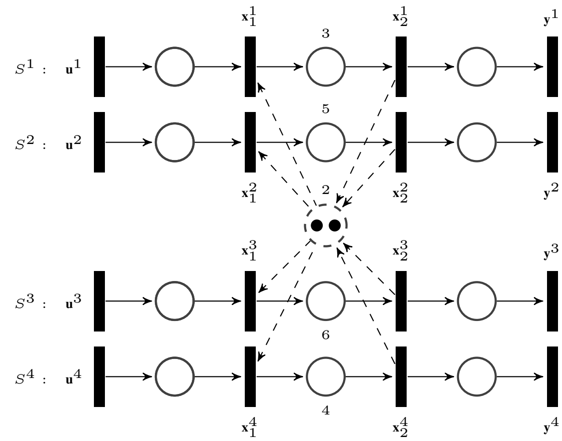

Fig. 4: 2 resources shared between 4 users.

t	0	1	2	3	4	5	6	$\cdots$	$+\infty$
u^1	e	1	3	3	6	6	6	$\cdots$	6
u^2	e	2	3	3	4	4	4	$\cdots$	4
u^3	e	3	3	3	5	5	5	$\cdots$	5
u^4	e	2	2	2	2	2	7	$\cdots$	7

Tab. 1 The input signals of Example 2

By considering the resource sharing phenomenon, the system becomes:

$$\begin{aligned} &S^1 : x^1(t) = A^1_{\alpha(t)} \otimes x^1(t-1) \oplus A^1_2 \otimes x^1(t-2) \\ &\qquad \oplus A^1_3 \otimes x^1(t-3) \oplus B^1_0 u^1(t), \\ &S^2 : x^2(t) = A^2_{\alpha(t)} \otimes x^2(t-1) \oplus A^2_2 \otimes x^2(t-2) \\ &\qquad \oplus A^2_5 \otimes x^2(t-5) \oplus B^2_0 u^2(t), \\ &S^3 : x^3(t) = A^3_{\alpha(t)} \otimes x^2(t-1) \oplus A^3_2 \otimes x^3(t-2) \\ &\qquad \oplus A^3_6 \otimes x^3(t-6) \oplus B^3_0 u^3(t), \\ &S^4 : x^4(t) = A^4_{\alpha(t)} \otimes x^4(t-1) \oplus A^4_2 \otimes x^4(t-2) \\ &\qquad \oplus A^4_4 \otimes x^4(t-4) \oplus B^4_0 u^4(t), \\ &\text{and} \quad y^k(t) = C \otimes x^k(t), \quad k = 1, ..., 4 \end{aligned} \qquad (10)$$

where $A^k_{\alpha(t)} = \begin{pmatrix} \alpha^k(t) & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}, k = 1, ..., 4$, and

$$\begin{aligned} &\alpha^1(t) = 2 \otimes (x^1_2(t-2) \oslash x^1_1(t-1)) \otimes (x^2_2(t-2) \oslash x^2_1(t-1)) \\ &\qquad \otimes (x^3_2(t-2) \oslash x^3_1(t-1)) \otimes (x^4_2(t-2) \oslash x^4_1(t-1)), \\ &\alpha^2(t) = \alpha^1(t) \otimes (x^1_1(t-1) \oslash x^1_1(t)), \\ &\alpha^3(t) = \alpha^2(t) \otimes (x^2_1(t-1) \oslash x^2_1(t)), \\ &\alpha^4(t) = \alpha^3(t) \otimes (x^3_1(t-1) \oslash x^3_1(t)). \end{aligned} \qquad (11)$$

With the resulting model it is straightforward to simulate the behavior of the system. For example, consider the input given by Tab. 1. According to Tab. 1, at time $t = 4$ for example, input transitions $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$, and $\mathbf{u}^4$ have fired 6, 4, 5, and 2 times, respectively. The corresponding series in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ are:

$$\begin{aligned} u^1 &= e\delta^0 \oplus 1\delta^1 \oplus 3\delta^3 \oplus 6\delta^{+\infty}, \\ u^2 &= e\delta^0 \oplus 2\delta^1 \oplus 3\delta^3 \oplus 4\delta^{+\infty}, \\ u^3 &= e\delta^0 \oplus 3\delta^3 \oplus 5\delta^{+\infty}, \\ u^4 &= e\delta^0 \oplus 2\delta^5 \oplus 7\delta^{+\infty}. \end{aligned} \tag{12}$$

Using available simulation tools, the system will respond as follows:

$$\begin{aligned} x_1^1 &= e\delta^0 \oplus 1\delta^5 \oplus 2\delta^7 \oplus 3\delta^{10} \oplus 4\delta^{12} \oplus 5\delta^{15} \oplus 6\delta^{+\infty}, \\ x_1^2 &= e\delta^0 \oplus 1\delta^{17} \oplus 2\delta^{20} \oplus 3\delta^{24} \oplus 4\delta^{+\infty}, \\ x_1^3 &= e\delta^{27} \oplus 1\delta^{31} \oplus 2\delta^{35} \oplus 3\delta^{39} \oplus 4\delta^{43} \oplus 5\delta^{+\infty}, \\ x_1^4 &= e\delta^{47} \oplus 1\delta^{51} \oplus 2\delta^{53} \oplus 3\delta^{57} \oplus 4\delta^{59} \oplus 5\delta^{63} \\ &\quad \oplus 6\delta^{65} \oplus 7\delta^{+\infty}. \end{aligned} \tag{13}$$

Note that these power series capture the information on when resources are allocated to the respective subsystem.

IV. Optimal feedforward control

The goal of this section is to calculate optimal control ensuring the just-in-time behavior with respect to output references z^k for each system S^k. We assume that $\exists t_f$ such that $z^k(t) = z^k(t_f) \forall t \geq t_f$, i.e., each z^k can be completely specified over a finite horizon. We first recall optimal control computation for the system $\tilde{S}^k$ (without RS)(see [1]).

A. Optimal control of $\tilde{S}^k$

An input u_*^k is optimal, if it is the greatest input satisfying

$$y^k \preceq z^k, \tag{14}$$

where "greatest" and $\preceq$ are to be interpreted in the sense of the order in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$. Hence, in conventional algebra, the optimal input corresponds to the least number of firings of the input transition that will ensure that the output transition fires at least $z^k(t)$ times (at any instant of time). This is also referred to as just-in-time control. It can be calculated as follows:
From (5) one can write:

$$\forall t \begin{cases} A_1^k x^k(t-1) \preceq x^k(t) \\ \vdots \\ A_{T_k}^k x^k(t-T_k) \preceq x^k(t) \\ B_0^k \otimes u^k(t) \preceq x^k(t) \\ \vdots \\ B_{M_k}^k \otimes u^k(t-M_k) \preceq x^k(t) \end{cases} \tag{15}$$

According to the results from Section II-B, (15) is equivalent to:

$$\forall t \begin{cases} x^k(t-1) \preceq A_1^k \circ\!\!\!\backslash\, x^k(t) \\ \vdots \\ x^k(t-T_k) \preceq A_{T_k}^k \circ\!\!\!\backslash\, x^k(t) \\ u^k(t) \preceq B_0^k \circ\!\!\!\backslash\, x^k(t) \\ \vdots \\ u^k(t-M_k) \preceq B_{M_k}^k \circ\!\!\!\backslash\, x^k(t) \end{cases} \tag{16}$$

(16) is equivalent to:

$$\forall t \begin{cases} x^k(t) \preceq A_1^k \circ\!\!\!\backslash\, x^k(t+1) \\ \vdots \\ x^k(t) \preceq A_{T_k}^k \circ\!\!\!\backslash\, x^k(t+T_k) \\ u^k(t) \preceq B_0^k \circ\!\!\!\backslash\, x^k(t) \\ \vdots \\ u^k(t) \preceq B_{M_k}^k \circ\!\!\!\backslash\, x^k(t+M_k). \end{cases} \tag{17}$$

Since, $y^k(t) = C^k x^k(t)$, from (14), we obtain:

$$x^k(t) \preceq C^k \circ\!\!\!\backslash\, z^k(t). \tag{18}$$

(17) and (18) can be summarized as:

$$x^k(t) \preceq A_1^k \circ\!\!\!\backslash\, x^k(t+1) \wedge ... \wedge A_{T_k}^k \circ\!\!\!\backslash\, x^k(t+T_k) \wedge C^k \circ\!\!\!\backslash\, z^k(t) \tag{19a}$$

$$u^k(t) \preceq B_0^k \circ\!\!\!\backslash\, x^k(t) \wedge ... \wedge B_{M_k}^k \circ\!\!\!\backslash\, x^k(t+M_k). \tag{19b}$$

Obviously, the greatest solution satisfying (19a) and (19b) is given by:

$$\zeta^k(t) = A_1^k \circ\!\!\!\backslash\, \zeta^k(t+1) \wedge ... \wedge A_{T_k}^k \circ\!\!\!\backslash\, \zeta^k(t+T_k) \wedge C^k \circ\!\!\!\backslash\, z^k(t), \tag{20}$$

where $\zeta^k(t) = \zeta^k(t_f) = C^k \circ\!\!\!\backslash\, z^k(t_f), \forall t \geq t_f$, and

$$u_*^k(t) = B_0^k \circ\!\!\!\backslash\, \zeta^k(t) \wedge ... \wedge B_{M_k}^k \circ\!\!\!\backslash\, \zeta^k(t+M_k). \tag{21}$$

Note that $\zeta^k(t)$ represents the vector of the least number of firings of all internal transitions up to time t such that (14) holds. $\zeta^k(t)$ is sometimes called the co-state of the system.

B. Optimal control with RS

We now consider an optimal control problem for the system (7) and (8).
First, the optimal control for subsystem S^1 is computed. Then the optimal control of system S^2 is computed under the restriction that the optimal behavior of S^1 is unchanged. This is repeated until the control for the lowest priority subsystem is calculated. In other words, the control of each subsystem will not decrease the performance of the higher priority subsystems. This can be conveniently expressed using the following *lexicographic* order on $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^m$: For $u = (u^1, u^2), u' = (u'^1, u'^2) \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^m$

$$u \preceq_{\mathcal{L}} u' \Leftrightarrow \begin{cases} u^1 \preceq u'^1, u^1 \neq u'^1 \quad \text{or} \\ u^1 = u'^1 \quad \text{and} \quad u^2 \preceq u'^2 \end{cases}, \tag{22}$$

where $\preceq$ is the previously introduced order in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$. Then, we look for the greatest $u = (u^1, ..., u^m)$ (in the above lexicographic order) such that $y^k(t) \preceq z^k(t), k = 1, ..., m$.

Proposition 1: The optimal solution for the above control problem is given by:

$$u_*^k(t) = B_0^k \circ\!\!\!\backslash\, \zeta^k(t) \wedge ... \wedge B_{M_k}^k \circ\!\!\!\backslash\, \zeta^k(t+M_k). \tag{23}$$

where

$$\begin{aligned} \zeta^k(t) = (&\textstyle\bigwedge_{l=1}^{T_k} (A_l^k \circ\!\!\!\backslash\, \zeta^k(t+l)) \wedge C^k \circ\!\!\!\backslash\, z^k(t) \wedge \\ &A_\gamma^k(t+d) \circ\!\!\!\backslash\, \zeta^k(t+d-1)) \oplus \zeta^k(t+1), \end{aligned} \tag{24}$$

with

$$(A^k_\gamma(t))_{i,j} = \begin{cases} \gamma^k(t) & \text{if } x^k_i = I^k \text{ and } x^k_j = O^k \\ \varepsilon & \text{else} \end{cases} \quad (25)$$

and $\gamma^k(t)$ is given by:

$$\begin{cases} \gamma^k(t) = \bigotimes_{r=1}^{k-1} (O^r_*(t-d) \oint I^r_*(t)) \otimes n \\ \qquad \text{for} \quad k = 2, ..., m \quad \text{and} \\ \gamma^1(t) = n. \end{cases} \quad (26)$$

As before, $\zeta^k(t) = \zeta^k(t_f) = C^k \lozenge z^k(t_f), \forall t \geq t_f$.

O^r_* and I^r_* are the corresponding entries of the optimal counter x^r_* previously computed using the optimal input $u^r_*, 1 \leq r \leq k-1$.

Proof: First, the optimal control of S^1 is computed by neglecting all the other subsystems. The computation of the optimal control u^1_* is then straightforward, using (20) and (21) for $k = 1$.

$$\begin{cases} \zeta^1(t) = A^1_1 \lozenge \zeta^1(t+1) \wedge ... \wedge A^1_{T_1} \lozenge \zeta^1(t+T_1) \wedge C^1 \lozenge z^1(t) \\ u^1_*(t) = B^1_0 \lozenge \zeta^1(t) \wedge ... \wedge B^1_{M_1} \lozenge \zeta^1(t+M_1) \end{cases} \quad (27)$$

and

$$x^1_*(t) = \bigoplus_{j=1}^{T_1} (A^1_j x^1_*(t-j)) \oplus \bigoplus_{j=0}^{M_1} (B^1_j u^1_*(t-j)) \quad (28)$$

Note that $\gamma^1(t) = n$, and therefore, $A^1_d \succeq A^1_\gamma$, hence $A^1_d \lozenge \zeta^1(t+d) \preceq A^1_\gamma \lozenge \zeta^1(t+d) \preceq A^1_\gamma \lozenge \zeta^1(t+d-1)$ (where the latter inequality is true because of ζ being nonincreasing). Therefore, as $d \leq T_1$,

$$\begin{aligned} & A^1_1 \lozenge \zeta^1(t+1) \wedge ... \wedge A^1_{T_1} \lozenge \zeta^1(t+T_1) \wedge C^1 \lozenge z^1(t) = \\ & A^1_1 \lozenge \zeta^1(t+1) \wedge ... \wedge A^1_{T_1} \lozenge \zeta^1(t+T_1) \wedge C^1 \lozenge z^1(t) \wedge \\ & \quad A^1_\gamma \lozenge \zeta^1(t+d-1) \end{aligned} \quad (29)$$

Because of being nonincreasing, this is the same as the right hand side of (24). In the next step, the optimal control for S^2 under the condition that the optimal behavior of S^1 is preserved is calculated. Recall that the evolution of the system with resource sharing is given by (7) and (8). Since the priority of S^2 is greater than the one of S^k for $k = 3, ..., m$, in order to compute the optimal control of S^2, all the systems with lower priority are neglected. In this case, the number of available resources at each instant of time for S^1 is denoted $\beta^1(t)$ and can be calculated by:

$$\beta^1(t) = O^1_*(t-d) \oint I^1_*(t-1) \otimes O^2(t-d) \oint I^2(t-1) \otimes n. \quad (30)$$

Note that $\beta^1(t)$ is equivalent to $\alpha^1(t)$ in (6) under the condition that S^1 is working under the optimal control u^1_*, and S^k for $k = 3, ..., m$ is neglected. As S^2 may not degrade the performance of S^1, the solution of (7), (8) for $k = 1$ and $\alpha^1 = \beta^1$ is x^1_*, i.e.,

$$\begin{aligned} x^1_*(t) = & (A^1_1 \oplus A^1_\beta(t)) x^1_*(t-1) \oplus ... \oplus A^1_{T_1} x^1_*(t-T_1) \oplus \\ & \bigoplus_{j=0}^{M_1} (B^1_j u^1_*(t-j)) \end{aligned} \quad (31)$$

where

$$(A^1_\beta(t))_{i,j} = \begin{cases} \beta^1(t) & \text{if } i = j \text{ and } x^1_i = I^1 \\ \varepsilon & \text{else.} \end{cases} \quad (32)$$

For (28) and (31) to be equivalent, we require

$$x^1_*(t) \succeq A^1_\beta(t) \otimes x^1_*(t-1) \quad (33)$$

which is equivalent to:

$$I^1_*(t) \succeq \beta^1(t) \otimes I^1_*(t-1) \quad (34)$$

Inserting (30) into (34) leads to:

$$I^1_*(t) \succeq O^1_*(t-d) \oint I^1_*(t-1) \otimes O^2(t-d) \oint I^2(t-1) \otimes n \otimes I^1_*(t-1) \quad (35)$$

Recalling the fact that in $\overline{\mathbb{Z}}_{min}$, $\oint$ corresponds to subtraction in the standard algebra, (35) leads to

$$I^1_*(t) \succeq O^1_*(t-d) \otimes O^2(t-d) \oint I^2(t-1) \otimes n,$$

or, equivalently,

$$I^2(t-1) \succeq O^1_*(t-d) \oint I^1_*(t) \otimes n \otimes O^2(t-d). \quad (36)$$

From (26),

$$\gamma^2(t) = O^1_*(t-d) \oint I^1_*(t) \otimes n.$$

Then,

$$I^2(t-1) \succeq \gamma^2(t) \otimes O^2(t-d) \quad (37)$$

Because of (25), (37) can be written in the following form:

$$x^2(t-1) \succeq A^2_\gamma(t) \otimes x^2(t-d). \quad (38)$$

From (38), we obtain $x^2(t-d) \preceq A^2_\gamma(t) \lozenge x^2(t-1)$ or, equivalently,

$$x^2(t) \preceq A^2_\gamma(t+d) \lozenge x^2(t+d-1). \quad (39)$$

Hence, preserving the optimal behavior of S^1 while running S^2 is guaranteed if (39) holds. On the other hand, the following constraint must also hold for S^2 (see(19a)):

$$x^2(t) \preceq A^2_1 \lozenge x^2(t+1) \wedge ... \wedge A^2_{T_2} \lozenge x^2(t+T_2) \wedge C^2 \lozenge z^2(t). \quad (40)$$

(39) and (40) lead to:

$$x^2(t) \preceq \bigwedge_{l=1}^{T_2} (A^2_l \lozenge x^2(t+l)) \wedge C^2 \lozenge z^2(t) \wedge A^2_\gamma(t+d) \lozenge x^2(t+d-1) \quad (41)$$

As $x^2(t)$ must be a nonincreasing function, the following constraint also holds:

$$x^2(t) \succeq x^2(t+1). \quad (42)$$

The greatest solution of (41) which satisfies (42) is denoted $\zeta^2(t)$ and satisfies:

$$\begin{aligned} \zeta^2(t) = & (\textstyle\bigwedge_{l=1}^{T_2} (A^2_l \lozenge \zeta^2(t+l)) \wedge C^2 \lozenge z^2(t) \wedge \\ & A^2_\gamma(t+d) \lozenge \zeta^2(t+d-1)) \oplus \zeta^2(t+1) \end{aligned} \quad (43)$$

and the optimal control for S^2 is given by:

$$u^2_*(t) = B^2_0 \lozenge \zeta^2(t) \wedge ... \wedge B^2_{M_2} \lozenge \zeta^2(t+M_2). \quad (44)$$

Iterating this procedure over subsystems with lower priority, i.e., $k = 3, ..., m$, results in (24). ■

Example 3: Consider the system (10) shown in Fig. 4. The reference signals are given by:

$$\begin{aligned} z^1 &= e\delta^{42} \oplus 1\delta^{46} \oplus 3\delta^{54} \oplus 6\delta^{+\infty}, \\ z^2 &= e\delta^{39} \oplus 1\delta^{50} \oplus 2\delta^{54} \oplus 3\delta^{+\infty}, \\ z^3 &= e\delta^{48} \oplus 1\delta^{50} \oplus 2\delta^{54} \oplus 4\delta^{+\infty}, \\ z^4 &= e\delta^{54} \oplus 2\delta^{+\infty}. \end{aligned}$$

According to (23) to (26), the optimal control for S^1 (the subsystem with the highest priority) is given by:

$$\begin{cases} \zeta^1(t) = A_2^1 \lozenge\zeta^1(t+2) \wedge A_3^1 \lozenge\zeta^1(t+3) \wedge C^1 \lozenge z^1(t) \\ u_*^1 = B_0^1 \lozenge\zeta^1(t), \end{cases}$$

which leads to:

$$u_*^1 = e\delta^{38} \oplus 1\delta^{41} \oplus 2\delta^{43} \oplus 3\delta^{46} \oplus 4\delta^{51} \oplus 6\delta^{+\infty}$$

The corresponding optimal evolution of S^1 can then be computed by:

$$\begin{cases} x_*^1(t) = A_2^1 \otimes x_*^2(t-2) \oplus A_3^1 \otimes x_*^1(t-3) \oplus B_0^1 \otimes u_*^1(t) \\ y_*^1(t) = C^1 \otimes x_*^1(t), \end{cases}$$

resulting in

$$y_*^1 = e\delta^{41} \oplus 1\delta^{44} \oplus 2\delta^{46} \oplus 3\delta^{49} \oplus 4\delta^{54} \oplus 6\delta^{+\infty} \preceq z_1$$

From the optimal behavior of S^1, $\gamma^2(t)$ can be calculated as:

$$\gamma^2(t) = x_{2*}^1(t-2) \not\phi x_{1*}^1(t) \otimes 2.$$

The optimal control for S_2 is given by:

$$\begin{cases} \zeta^2(t) = (A_2^2 \lozenge\zeta^2(t+2) \wedge A_5^2 \lozenge\zeta^2(t+5) \\ \qquad \wedge A_\gamma^2(t+2) \lozenge\zeta^2(t+1) \wedge C^2 \lozenge z_2(t)) \oplus \zeta^2(t+1) \\ u_*^2(t) = B_0^2 \lozenge\zeta^2(t) \end{cases}$$

with $A_\gamma^2(t) = \begin{pmatrix} \varepsilon & \gamma^2(t) \\ \varepsilon & \varepsilon \end{pmatrix}$. This leads to:

$$\begin{aligned} u_*^2 &= e\delta^{27} \oplus 1\delta^{31} \oplus 2\delta^{34} \oplus 3\delta^{+\infty}, \\ y_*^2 &= e\delta^{32} \oplus 1\delta^{36} \oplus 2\delta^{39} \oplus 3\delta^{+\infty} \preceq z^2. \end{aligned}$$

This procedure is repeated to calculate control for $k = 3, 4$. For $k = 3$

$$\gamma^3(t) = x_{2*}^2(t-2) \not\phi x_{1*}^2(t) \otimes x_{2*}^1(t-2) \not\phi x_{1*}^1(t) \otimes 2,$$

and the control for S^3 is given by:

$$\begin{cases} \zeta^3(t) = (A_2^3 \lozenge\zeta^3(t+2) \wedge A_6^3 \lozenge\zeta^3(t+6) \wedge \\ \qquad A_\gamma^3(t+2) \lozenge\zeta^3(t+1) \wedge C^3 \lozenge z^3(t)) \oplus \zeta^3(t+1) \\ u_*^3(t) = B_0^3 \lozenge\zeta^3(t) \end{cases}$$

with $A_\gamma^3(t) = \begin{pmatrix} \varepsilon & \gamma^3(t) \\ \varepsilon & \varepsilon \end{pmatrix}$. This leads to:

$$\begin{aligned} u_*^3 &= e\delta^{11} \oplus 1\delta^{15} \oplus 2\delta^{18} \oplus 3\delta^{23} \oplus 4\delta^{+\infty}, \\ y_*^3 &= e\delta^{17} \oplus 1\delta^{21} \oplus 2\delta^{24} \oplus 3\delta^{29} \oplus 4\delta^{+\infty} \preceq z^3, \end{aligned}$$

For $k = 4$, we get:

$$\begin{aligned} \gamma^4(t) = x_{2*}^3(t-2) \not\phi x_{1*}^3(t) \otimes x_{2*}^2(t-2) \not\phi x_{1*}^2(t) \otimes \\ x_{2*}^1(t-2) \not\phi x_{1*}^1(t) \otimes 2, \end{aligned}$$

and the control for S^4 is given by:

$$\begin{cases} \zeta^4(t) = (A_2^4 \lozenge\zeta^4(t+2) \wedge A_4^4 \lozenge\zeta^4(t+4) \wedge \\ \qquad A_\gamma^4(t+2) \lozenge\zeta^4(t+1) \wedge C^4 \lozenge z^4(t)) \oplus \zeta^4(t+1) \\ u_*^4(t) = B_0^4 \lozenge\zeta^4(t) \end{cases}$$

with $A_\gamma^4(t) = \begin{pmatrix} \varepsilon & \gamma^4(t) \\ \varepsilon & \varepsilon \end{pmatrix}$. This leads to:

$$\begin{aligned} u_*^4 &= e\delta^5 \oplus 1\delta^9 \oplus 2\delta^{+\infty}, \\ y_*^4 &= e\delta^9 \oplus 1\delta^{13} \oplus 2\delta^{+\infty} \preceq z^4 \end{aligned}$$

V. CONCLUSION

In this paper, we have discussed modeling and control of a class of timed Petri nets with resource sharing problems. In particular, several subsystems, each described by a timed event graph (TEG) compete for n resources modeled by tokens in a joint place. We assume a given prioritization policy and model the resulting system in the min-plus algebra. Under this policy, we provide optimal control (in the corresponding lexicographical order) for the overall system. In particular, the number of firings of the input transition of the top priority subsystem is, at any instant of time, as small as possible while guaranteeing that the output transition fires at least as often as specified by a given reference signal. The subsystem with the k-th priority, $k \geq 2$, is subject to the same notion of optimality, but is restrained by the temporal evolution of the $k-1$ subsystems with higher priority. Although, for notational simplicity, we discussed our modeling and control approach for the case where the tokens of only one place are shared, and where only one transition in each subsystem has an incoming (outcoming) arc from (to) the shared place, this can be readily translated to more general scenarios.

REFERENCES

[1] F. Baccelli and G. Cohen and G. J. Olsder and J.-P. Quadrat. Synchronization and linearity: an algebra for discrete event systems. Wiley, New York, 1992.

[2] B. De Schutter and T. van den Boom. Model predictive control for max-plus-linear systems. Proceedings of the 2000 American Control Conference, Chicago, Illinois, 4046-4050, 2000.

[3] T.J.J Van den Boom and B. De Schutter. Modeling and control of switching max-plus-linear systems with random and deterministic switching . Discrete Event Dynamic Systems: Theory and Applications, 22(3), pp. 293-332, 2012.

[4] M. Alsaba, S. Lahaye, and J. L. Boimond. On just in time Control of switching Max-Plus Linear Systems. In Proceedings of ICINCO?06, Setubal, Portugal, pp. 79-84, 2006.

[5] A. Correia, A. Abbas-Turki, R. Bouyekhf, and A.E. Moudni. A dioid model for invariant resource sharing problems. IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans, Chicago, 39(4), 770-781, 2009.

[6] B. Addad, S. Amari, J. Lesage. Linear Time-Varying (Max,+) Representation of Conflicting Timed Event Graphs. 10th Int Workshop on Discrete Event Systems, Aug. 2010, Berlin, Germany. pp. 310-315.

[7] X. Allamigon, V. Boeuf, and S. Gaubert. Performance evaluation of an emergency call center: Tropical polynomial systems applied to timed Petri nets. InFORMATS'15, V 9268, Lecture Notes in Computer Science, Springer, 2015.

[8] S. Yoshida, H. Takahashi and h. Goto. Resolution of Resource Conflict for a Single Project in Max-plus Linear Representation. Journal of Computation Modelling, vol. 1(1), pp. 33-47, 2011.

[9] B. Heidergott, G. J. Olsder and J. van der Woude. Max Plus at work : modeling and analysis of synchronized systems : a course on Max-Plus algebra and its applications. Princeton University Press, Princeton (N.J.), 0-691-11763-2, 2006.

[10] G. Cohen and P. Moller and J.-P. Quadrat and M. Viot. (1989). Algebraic Tools for the Performance Evaluation of Discrete Event Systems. P-IEEE, 77(1), 39-58, 1989.

[11] T.S. Blyth and M.F. Janowitz. Residuation Theory. Pergamon Press, Oxford, 1972.

[12] G. Cohen. Residuation and Applications. Algèbres Max-Plus et applications en informatique et automatique. INRIA, Noirmoutier, May (26), 1998.

[13] L. Hardouin, B. Cottenceau, S. Lagrange, E. Le Corronc. Performance Analysis of Linear Systems over Semiring with Additive Inputs. Worksop On Discrete Event Systems, WODES 08, Göteborg, May 2008.

[14] L. Hardouin, B. Cottenceau, and M. Lhommeau. Software tools for manipulating periodic series. `Available at istia.univ-angers.fr/ hardouin/outils.html.` 2001.

[15] B. Cottenceau, L. Hardouin, M. Lhommeau, and Boimond. Data processing tool for calculation in dioid. In Proceedings of the 5th International Workshop on Discrete Event Systems, WODES, 2000. Ghent, Belgium. `Available at istia.univ-angers.fr/ hardouin/outils.html.`