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Model Decomposition of Weight-Balanced Timed Event Graphs in Dioids: Application to Control Synthesis

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Abstract: For Timed Event Graphs (TEGs), model reference control is a well known approach to stabilize and to improve the performance of a system. This method is based on dioid and residuation theory. In this work, we study model reference control for the class of single-input and single-output (SISO) Weight-Balanced Timed Event Graphs (WBTEGs), which is an extension of TEGs and exhibits event-variant behavior. By modeling the behavior of WBTEGs in a dioid structure, we propose a decomposition of the dynamic behavior into an event-variant and an event-invariant part. We further show that the event-variant part is invertible, hence the problem of model reference control for WBTEGs can be reduced to the case of conventional TEGs.

Keywords: Dioid, Weighted Timed Event Graph (WTEG), controller synthesis, discrete-event systems.

1. INTRODUCTION AND MOTIVATION

TEGs are a subclass of timed Petri nets where each place has exactly one input and one output transition and all arcs have weight 1. Weighted Timed Event Graphs (WTEGs) are an extension of TEGs where the weights on the arcs can take values in $\mathbb{N} = \{1, 2, \cdots\}$. This formalism is popular to model systems ruled by synchronization, such as manufacturing processes or transport networks. The weights associated with arcs in WTEGs are suitable to express batch/split processes, for instance, when several occurrences of events are needed to induce a following event or when one event can result in several following events. This leads to an event-variant behavior which cannot be expressed by conventional TEGs. An equivalent graphical representation for WTEGs are Synchronous Data-Flow (SDF) graphs, and homogeneous SDF graphs are an equivalent representation of TEGs. WTEGs and SDF graphs have been widely studied e.g., for WTEGs see Marchetti and Munier-Kordon (2010), Teruel et al. (1992), and for SDF graphs see Sriram and Bhattacharyya (2000). SDF graphs are a popular framework for modeling and analyzing real-time embedded and multiprocessor systems. Most of the related work focuses on throughput analysis, which is a key property for the performance of those systems. For TEGs (resp. homogeneous SDF graphs), many tools have been developed for performance evaluation, for instance, their behavior can be modeled as linear state space representations in a tropical algebra structure called (max,+) algebra. In Baccelli et al. (1992) and Cochet-Terrasson et al. (1998), spectral analysis for linear (max,+) systems is introduced, which has an application in throughput calculation for TEGs. However, these methods cannot be directly applied to WTEGs. Munier (1993) suggests a transformation, which maps subclass of WTEGs into equivalent TEGs. A similar transformation is known for SDF graphs, see Sriram and Bhattacharyya (2000). However, the transformation can be computationally quite expensive, since the transformation significantly increases the number of transitions in the corresponding TEG.

Recently, Cottenceau et al. (2014) introduced a dioid, denoted $\mathcal{E}^*[\![\delta]\!]$, for modeling and analysis of an important subclass of WTEGs - the class of WTEGs where parallel paths have balanced weights. This class is therefore called Weight-Balanced Timed Event Graphs (WBTEGs). The dioid $\mathcal{E}^*[\![\delta]\!]$ is composed of four basic operators, event shift γ, time shift δ, event multiplication (split) μ and event division (batch) β. $\mathcal{E}^*[\![\delta]\!]$ gives a natural way to model a WBTEG in a state space representation, similar to the modeling process of TEGs in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, see Baccelli et al. (1992). Furthermore, it is shown that the input-output behavior of WBTEGs can be described by ultimately periodic series in $\mathcal{E}^*[\![\delta]\!]$. To obtain these transfer functions, the Kleene star operation for elements in $\mathcal{E}^*[\![\delta]\!]$ plays a key role. In this work, we show how elements in the dioid $\mathcal{E}^*[\![\delta]\!]$ can be decomposed into an event-variant part and an event-invariant core given by a matrix in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. Furthermore, we show that all basic operations on the dioid $\mathcal{E}^*[\![\delta]\!]$, can be reduced to operations between matrices in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. An advantage of this method is that the existing tools for performance evaluation and controller synthesis for conventional TEGs can be used for WBTEGs. Furthermore, we present a new algorithm for the calculation of the Kleene star for elements in $\mathcal{E}^*[\![\delta]\!]$, which is in general computationally more efficient than the algorithm presented in Cottenceau et al. (2014).

The paper is organized as follows: Section 2 summarizes the necessary facts on Petri nets, WBTEG and dioid theory. In Section 3, the modeling process of a WBTEGs in the dioid $\mathcal{E}^*[\![\delta]\!]$ is recalled. Section 4 introduces a decomposition method for elements in $\mathcal{E}^*[\![\delta]\!]$. Moreover, algorithms for the Kleene star

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calculation in $\mathcal{E}^*[\![\delta]\!]$ are compared. Finally, in Section 5, the controller design process for WBTEGs is illustrated.

2. WEIGHTED TIMED EVENT GRAPHS AND DIOIDS

2.1 Petri nets and Timed Event Graphs

In the following, we restate the necessary facts on Petri nets and TEGs (see, e.g., Baccelli et al. (1992) Cassandras and Lafortune (1999)). Matrices and vectors are indicated by bold letters. A Petri net graph is a directed bipartite graph $\mathcal{N} = (P, T, w)$, where:

- $P = \{p_1, p_2, \ldots, p_n\}$ is the finite set of places.
- $T = \{t_1, t_2, \ldots, t_m\}$ is the finite set of transitions.
- $w : (P \times T) \cup (T \times P) \to \mathbb{N}_0$ is the weight function.

$A := \{(p_i, t_j)|w(p_i, t_j) > 0\} \cup \{(t_j, p_i)|w(t_j, p_i) > 0\}$ is the arc set of the Petri net graph $\mathcal{N}$. A Petri net consists of a Petri net graph $\mathcal{N}$ and a vector of initial markings $\boldsymbol{M}_0 \in (\mathbb{N}_0)^n$, i.e. an initial distribution of tokens over places in $\mathcal{N}$. ${}^\bullet t_j := \{p_i \in P|(p_i, t_j) \in A\}$ is the set of input places of transition t_j and $t_j^\bullet := \{p_i \in P|(t_j, p_i) \in A\}$ is the set of output places of transition t_j. Conversely, ${}^\bullet p_i := \{t_j \in T|(t_j, p_i) \in A\}$ is the set of input transitions of p_i and $p_i^\bullet := \{t_j \in T|(p_i, t_j) \in A\}$ is the set of output transitions of place p_i. A transition t_j can fire, if and only if the number of tokens $(\boldsymbol{M})_i$ of all input places p_i is at least $w(p_i, t_j)$. When a transition t_j fires, the marking in place p_i changes according to

$$(\boldsymbol{M'})_i = (\boldsymbol{M})_i + w(t_j, p_i) - w(p_i, t_j)\,,$$

where $(\boldsymbol{M})_i$ and $(\boldsymbol{M'})_i$ are the marking of the place p_i before and after the firing of transition t_j. A timed Petri net with holding times is a triple $(\mathcal{N}, \boldsymbol{M}_0, \boldsymbol{\phi})$, where $(\mathcal{N}, \boldsymbol{M}_0)$ is a Petri net and $\boldsymbol{\phi} \in (\mathbb{N}_0)^n$ represents the holding times of the places, i.e., $(\boldsymbol{\phi})_i$ is the time a token has to remain in place p_i before it contributes to the firing of a transition in $p_i^\bullet$.

2.2 Weighted Timed Event Graphs

Definition 1. A Petri net $(\mathcal{N}, \boldsymbol{M}_0)$ is called Weighted Event Graph, if every place has exactly one input and one output transition, i.e., $\forall p_i \in P : |p_i^\bullet| = |{}^\bullet p_i| = 1$. ◁

Definition 2. A timed Petri net $(\mathcal{N}, \boldsymbol{M}_0, \boldsymbol{\phi})$ is called a Weighted Timed Event Graph (WTEG) if $(\mathcal{N}, \boldsymbol{M}_0)$ is a Weighted Event Graph. ◁

A basic directed path $t_j \to p_i \to t_o$ of a WTEG is such that $t_j \in {}^\bullet p_i$ and $t_o \in p_i^\bullet$. As $|p_i^\bullet| = |{}^\bullet p_i| = 1\ \forall p_i \in P$, each place appears in precisely one basic directed path, which we will denote π_i. A directed path is a sequence of basic directed paths, $\pi = \pi_{i_1} \cdots \pi_{i_q}$, where $p_{i_j}^\bullet = {}^\bullet p_{i_{j+1}}, j = \{1, \cdots, q-1\}$. A WTEG is *strongly connected*, if $\forall t_j, t_l \in T$ there exists a directed path from t_j to t_l.

Definition 3. (Gain of a Path). The gain of a basic directed path is defined as

$$\Gamma(\pi_i) = \Gamma(t_j, p_i, t_o) = \frac{w(t_j, p_i)}{w(p_i, t_o)}. \qquad (1)$$

◁

The gain of a directed path $\pi = \pi_{i_1} \cdots \pi_{i_q}$ is then the product of all basic directed path gains, i.e., $\Gamma(\pi) = \prod_{j=1}^{q} \Gamma(\pi_{i_j})$. We can divide the set of transitions of a WTEG into input, output and internal transitions. Input transitions are transitions without input places. Output transitions are transitions without output places and all other transitions are called internal transitions. A SISO WTEG has exactly one input and one output transition. In addition, we define a WTEG as *stable*, if the number of tokens in every place is *bounded*. Note that this implies that *stable* WTEGs do not have input transitions. A Weight-Balanced Timed Event Graph (WBTEG) is a WTEG with the additional restriction that parallel paths, i.e., paths beginning and ending in the same transition must have the same gain (they are weight balanced).

Definition 4. (Earliest Functioning Rule). A WTEG is operating under the earliest functioning rule, if all internal and output transitions fire as soon as they are enabled. ◁

2.3 Dioid Theory

A dioid $\mathcal{D}$ is an algebraic structure with two binary operations, $\oplus$ (addition) and $\otimes$ (multiplication). Addition is commutative, associative and idempotent (i.e. $\forall a \in \mathcal{D},\ a \oplus a = a$). The neutral element for addition, denoted by ε, is absorbing for multiplication (i.e., $\forall a \in \mathcal{D},\ a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$). Multiplication is associative, distributive over addition and has a neutral element denoted by e. Both operations can be extended to the matrix case. For matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{D}^{m\times n}$, $\boldsymbol{C} \in \mathcal{D}^{n\times q}$ and a scalar $\lambda \in \mathcal{D}$, matrix addition and multiplication are defined by

$$(\boldsymbol{A} \oplus \boldsymbol{B})_{ij} := (\boldsymbol{A})_{ij} \oplus (\boldsymbol{B})_{ij}, \quad (\lambda \otimes \boldsymbol{A})_{ij} := \lambda \otimes (\boldsymbol{A})_{ij},$$

$$(\boldsymbol{A} \otimes \boldsymbol{C})_{ij} := \bigoplus_{k=1}^{n} \left((\boldsymbol{A})_{ik} \otimes (\boldsymbol{C})_{kj}\right).$$

Note that, as in conventional algebra, the multiplication symbol $\otimes$ is often omitted. A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. A complete dioid is a partially ordered set, with a canonical order $\succeq$ defined by $a \oplus b = a \Leftrightarrow a \succeq b$. The infimum operator can then be defined by $a, b \in \mathcal{D}\ a \wedge b = \bigoplus\{x \in \mathcal{D}\,|x \oplus a = a, x \oplus b = b\}$. On a complete dioid, the Kleene star of an element $a \in \mathcal{D}$, denoted a^*, is defined by $a^* = \bigoplus_{i=0}^{\infty} a^i$ with $a^0 = \mathrm{e}$ and $a^{i+1} = a \otimes a^i$. Since a complete dioid is closed for infinite sums, a^* exists and $a^* \in \mathcal{D}$.

Theorem 1. (Baccelli et al. (1992)). On a complete dioid $\mathcal{D}$, $x = a^*b$ is the least solution of the implicit equation $x = ax \oplus b$. ◁

Residuation theory is a formalism to address the problem of approximate mapping inversion over ordered sets, see Baccelli et al. (1992).

Definition 5. (Residuation). Let $\mathcal{F}$ and $\mathcal{L}$ be ordered sets and $f : \mathcal{F} \to \mathcal{L}$ an isotone mapping, i.e., $a \preceq b$ implies $f(a) \preceq f(b)$. The mapping f is said to be residuated if for all $y \in \mathcal{L}$, the least upper bound of the subset $\{x \in \mathcal{F}|f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f. ◁

Since a complete dioid is an ordered set, this notion is applicable for specific mappings defined over the dioid. For instance, on a complete dioid the mapping $R_a : x \mapsto xa$, (right multiplication) resp. $L_a : x \mapsto ax$ (left multiplication) are residuated. The residual mappings are denoted $R_a^\sharp(b) = b \phi a = \bigoplus\{x|xa \preceq b\}$ (right division by a) resp. $L_a^\sharp(b) = a \backslash\!\!\circ b = \bigoplus\{x|ax \preceq b\}$ (left division by a). In analogy to the extension of the product to the matrix case, we can extend left and right division to matrices with entries in a complete dioid.

2.4 Dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

TEGs can be conveniently modeled via two-dimensional power series in γ and δ with Boolean coefficients, where γ and δ can be interpreted as event-shift and time-shift operators, respectively. Such a series is then given by $s = \bigoplus_{i\in\mathbb{Z}} s(i)\gamma^{\nu_i}\delta^{\varsigma_i}$ in which $s(i) \in \mathbb{B} = \{\mathrm{\tilde{e}}, \tilde{\varepsilon}\}$. The set of these power series is denoted by $\mathcal{B}[\![\gamma,\delta]\!]$, and $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is the quotient dioid of $\mathcal{B}[\![\gamma,\delta]\!]$ induced by the equivalence relation $x\mathcal{R}y \Leftrightarrow \gamma^*(\delta^{-1})^*x = \gamma^*(\delta^{-1})^*y$ (Gaubert and Klimann (1991)). It is a complete dioid and the quotient structure induces the following simplification rules in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

$$\delta^{\varsigma} \oplus \delta^{\varsigma'} = \delta^{\max(\varsigma,\varsigma')}, \ \ \gamma^{\nu} \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\nu')}.$$

A monomial in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is given by $\gamma^{\nu}\delta^{\varsigma}$, a polynomial is a finite sum of monomials, e.g., $\gamma^{\nu_1}\delta^{\varsigma_1} \oplus \gamma^{\nu_2}\delta^{\varsigma_2} \oplus \gamma^{\nu_3}\delta^{\varsigma_3}$. The zero and unit element in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ are $\varepsilon = \bigoplus_{\nu,\varsigma\in\mathbb{Z}} \tilde{\varepsilon}\gamma^{\nu}\delta^{\varsigma}$ and $\mathrm{e} = \gamma^0\delta^0$, respectively. In Gaubert and Klimann (1991), it is shown, that the input-output behavior of a TEG can be described by rational series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. It is always possible to put these series into an ultimately periodic form, i.e., $s = p \oplus q(\gamma^{\nu}\delta^{\varsigma})^*$, in which p, q are polynomials in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. In the following, we give the basic results for calculations with periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. For further information, the reader is invited to consult Gaubert and Klimann (1991) and Baccelli et al. (1992).

Theorem 2. (Gaubert and Klimann (1991)). Let $s_1 = p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $s_2 = p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$ be two ultimately periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, where p_1, q_1, p_2, q_2 are polynomials in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and $\nu_1, \nu_2, \tau_1, \tau_2 \in \mathbb{N}$. Furthermore, $s_1 \neq \varepsilon$, $s_2 \neq \varepsilon$ and the asymptotic slope of s_1 is defined by $\sigma(s_1) = \tau_1/\nu_1$ (resp. $\sigma(s_2) = \tau_2/\nu_2$), then

- $s_1 \oplus s_2$ is an ultimately periodic series such that $\sigma(s_1 \oplus s_2) = \max(\sigma(s_1), \sigma(s_2))$.
- $s_1 \otimes s_2$ is an ultimately periodic series such that $\sigma(s_1 \otimes s_2) = \max(\sigma(s_1), \sigma(s_2))$.
- $(s_1)^*$ is an ultimately periodic series.

◁

3. WBTEGs AND THE DIOID $\mathcal{E}^*[\![\delta]\!]$

3.1 Dioids of Event Operators $\mathcal{E}$

The dioid $\mathcal{E}^*[\![\delta]\!]$ introduced in Cottenceau et al. (2014) is suitable to model dynamic phenomena occurring in WBTEGs. For the modeling process of WBTEGs in $\mathcal{E}^*[\![\delta]\!]$, a counter function $x_i : \mathbb{Z} \to \mathbb{Z}_{min}$, where $\mathbb{Z}_{min} = \mathbb{Z} \cup \{\infty, -\infty\}$, is associated to each transition t_i. $x_i(\xi)$ gives the accumulated number of events up to time ξ. A counter function is naturally a non-decreasing function, i.e. $x_i(\xi+1) \geq x_i(\xi)$, and the set of counter functions is denoted by Σ. On Σ addition $\oplus$ and multiplication $\otimes$ are defined as follows:

$$x, y \in \Sigma, \ (x \oplus y)(\xi) := \min(x(\xi), y(\xi)),$$
$$\lambda \in \mathbb{Z}_{min} \ (\lambda \otimes x)(\xi) := \lambda + x(\xi).$$

The $\oplus$ operation induces an order relation on Σ, i.e., for $x, y \in \Sigma$, $x \preceq y \Leftrightarrow x \oplus y = y$. An operator is a map $\mathcal{K} : \Sigma \to \Sigma$ which is linear if (a) $\forall x, y \in \Sigma : \mathcal{K}(x \oplus y) = \mathcal{K}(x) \oplus \mathcal{K}(y)$ and (b) $\lambda \otimes \mathcal{K}(x) = \mathcal{K}(\lambda \otimes x)$. An operator is additive if (a) is satisfied. Dynamic phenomena arising in WBTEGs can be described by the following additive basic operators:

$$\varsigma \in \mathbb{Z}, \ \delta^{\varsigma} : \forall x \in \Sigma, \ (\delta^{\varsigma}x)(\xi) = x(\xi - \varsigma), \tag{2}$$
$$\nu \in \mathbb{Z}, \ \gamma^{\nu} : \forall x \in \Sigma, \ (\gamma^{\nu}x)(\xi) = x(\xi) + \nu, \tag{3}$$
$$b \in \mathbb{N}, \ \beta_b : \forall x \in \Sigma, (\beta_b x)(\xi) = \lfloor x(\xi)/b \rfloor, \tag{4}$$
$$m \in \mathbb{N}, \ \mu_m : \forall x \in \Sigma, (\mu_m x)(\xi) = m \times x(\xi), \tag{5}$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to $a \in \mathbb{Q}$. The event-shift and time-shift operators γ and δ are linear, whereas μ_m, β_b for $m \neq 1, b \neq 1$, are only additive. The three operators $\{\gamma^{\nu}, \mu_m, \beta_b\}$ are essential to describe the event-variant behavior of WBTEGs. Therefore, in the following we discuss them in detail.

Definition 6. (Dioid of E-operators $\mathcal{E}$). We denote by $\mathcal{E}$ the dioid of operators obtained by sums and compositions of operators in $\{\gamma^{\nu}, \beta_b, \mu_m\}$ with $\nu \in \mathbb{Z}$, and $b, m \in \mathbb{N}$, equipped with addition and multiplication defined by, $\forall w_1, w_2 \in \mathcal{E}, \ x \in \Sigma$,

$$(w_1 \oplus w_2)(x) = w_1(x) \oplus w_2(x), \ (w_1 \otimes w_2)(x) = w_1(w_2(x)).$$

The identity operator (resp. zero element) is denoted by $\mathrm{e} : \forall x \in \Sigma, \ (\mathrm{e}(x))(\xi) = x(\xi)$ (resp. $\varepsilon : \forall x \in \Sigma, \ (\varepsilon(x))(\xi) = \infty$). ◁

Note that the operator δ^{ς} is not in $\mathcal{E}$. $\mathcal{E}$ is a complete dioid, and an element $w \in \mathcal{E}$ is called E-operator hereafter. Moreover the identity operator can be written as: $\mathrm{e} = \gamma^0 = \mu_1 = \beta_1$.

Proposition 1. (Cottenceau et al. (2014)). The basic E-operators $\{\gamma^{\nu}, \mu_m, \beta_b\}$ satisfy the following relations

$$\gamma^{\nu}\gamma^{\nu'} = \gamma^{\nu+\nu'}, \tag{6}$$
$$\mu_m\gamma^n = \gamma^{n\times m}\mu_m, \qquad \gamma^n\beta_b = \beta_b\gamma^{n\times b}. \tag{7}$$

Since E-operators only affect event numbering an E-operator w can be described by a Counter-value to Counter-value (C/C) function $\mathcal{F}_w : \mathbb{Z}_{min} \to \mathbb{Z}_{min}$. An input counter value $k_i = x(\xi)$ is mapped to an output counter value k_o, for instance $\mathcal{F}_{\mu_m\beta_b}(k_i) = \lfloor k_i/b \rfloor m$. This follows immediately from the definition of μ_m, β_b in (5),(4). There is an isomorphism between the set of E-operators and the set of (C/C) functions. The order relation over the dioid $\mathcal{E}$ corresponds to the order induced by the min operation on (C/C) functions. For $w_1, w_2 \in \mathcal{E}$

$$\begin{aligned} & w_1 \succeq w_2 \Leftrightarrow w_1 \oplus w_2 = w_1, \\ & \Leftrightarrow w_1(x) \oplus w_2(x) = w_1(x) \ \forall x \in \Sigma, \\ & \Leftrightarrow (w_1(x))(\xi) \oplus (w_2(x))(\xi) = (w_1(x))(\xi) \ \forall x \in \Sigma, \forall \xi \in \mathbb{Z}, \\ & \Leftrightarrow \mathcal{F}_{w_1}(k) \leq \mathcal{F}_{w_2}(k) \ \forall k \in \mathbb{Z}_{min}. \end{aligned} \tag{8}$$

Definition 7. (Periodic E-operators). An E-operator $w \in \mathcal{E}$ is called (m, b)-periodic if $\forall k \in \mathbb{Z}_{min}, \ \mathcal{F}_w(k+b) = m + \mathcal{F}_w(k)$, with $m, b \in \mathbb{N}$. ◁

The set of periodic E-operators is denoted by $\mathcal{E}_{per}$. The gain of an (m, b)-periodic E-operator $w \in \mathcal{E}_{per}$ is defined as $\Gamma(w) := m/b$. The basic E-operators $\mathrm{e}, \gamma^{\nu}, \beta_b$ and μ_m are periodic, with gain $\Gamma(\gamma^n) = \Gamma(\mathrm{e}) = 1$, $\Gamma(\beta_b) = 1/b$ and $\Gamma(\mu_m) = m$. Moreover, for $w_1, w_2 \in \mathcal{E}_{per}$ with $\Gamma(w_1) = \Gamma(w_2)$, we have $\Gamma(w_1 \oplus w_2) = \Gamma(w_1)$ and therefore $w_1 \oplus w_2 \in \mathcal{E}_{per}$. For $w_3, w_4 \in \mathcal{E}_{per}$, we have $\Gamma(w_3 \otimes w_4) = \Gamma(w_3) \times \Gamma(w_4)$ and therefore $w_3 \otimes w_4 \in \mathcal{E}_{per}$.

Proposition 2. (Cottenceau et al. (2014)). E-operators associated with WBTEGs are periodic.

The (m, b)-periodic $\mu_m\beta_b$ operator can be extended to a multiple of its period in the following way

$$\mu_m\beta_b = \bigoplus_{i=0}^{n-1} \gamma^{im}\mu_{nm}\beta_{nb}\gamma^{(n-1-i)b}. \quad (9)$$

For instance with $n = 3$, the operator $\mu_1\beta_2$ can be written as $\mu_3\beta_6\gamma^4 \oplus \gamma^1\mu_3\beta_6\gamma^2 \oplus \gamma^2\mu_3\beta_6$. Fig. 1 illustrates the extension of this $\mu_1\beta_2$ operator. The intersection of the areas beneath $\mathcal{F}_{\mu_3\beta_6\gamma^4}$, $\mathcal{F}_{\gamma^1\mu_3\beta_6\gamma^2}$ and $\mathcal{F}_{\gamma^2\mu_3\beta_6}$ is equal to area beneath the (C/C) function $\mathcal{F}_{\mu_1\beta_2}$.

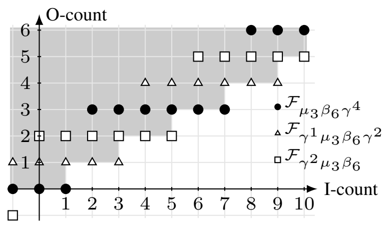

Fig. 1. $\min(\mathcal{F}_{\mu_3\beta_6\gamma^4}, \mathcal{F}_{\gamma^1\mu_3\beta_6\gamma^2}, \mathcal{F}_{\gamma^2\mu_3\beta_6})$ is equal to $\mathcal{F}_{\mu_1\beta_2}$.

Proposition 3. (Cottenceau et al. (2014)). Let $w \in \mathcal{E}$, then

$$\gamma^n \diamond\!\!\backslash w = \gamma^{-n}w, \qquad w \diamond\!\!/ \gamma^n = w\gamma^{-n}, \quad (10)$$

$$\mu_m \diamond\!\!\backslash w = \beta_m\gamma^{m-1}w, \qquad w \diamond\!\!/ \mu_m = w\beta_m, \quad (11)$$

$$\beta_b \diamond\!\!\backslash w = \mu_b w, \qquad w \diamond\!\!/ \beta_b = w\gamma^{b-1}\mu_b. \quad (12)$$

3.2 Dioid $\mathcal{E}^*[\![\delta]\!]$

E-operators commute with the time-shift operator δ^t, i.e., $\forall w \in \mathcal{E}$, $\delta^1 w = w\delta^1$, Cottenceau et al. (2014). Therefore, one can define formal power series in δ with exponents in $\mathbb{Z}$ and coefficients $w \in \mathcal{E}$ as follows.

Definition 8. (Dioid $\mathcal{E}^*[\![\delta]\!]$) We denote by $\mathcal{E}^*[\![\delta]\!]$ the quotient dioid in the set of formal power series in one variable δ with exponents in $\mathbb{Z}$ and coefficients in the non commutative complete dioid $\mathcal{E}$ induced by the equivalence relation $\forall s \in \mathcal{E}^*[\![\delta]\!]$, $s = (\gamma^1)^*s = s(\gamma^1)^* = (\delta^{-1})^*s = s(\delta^{-1})^*$. ◁

The subset of $\mathcal{E}^*[\![\delta]\!]$ obtained by restricting the coefficients to $\mathcal{E}_{per}$, i.e. the set of periodic operators, is denoted by $\mathcal{E}^*_{per}[\![\delta]\!]$. A monomial in $\mathcal{E}^*_{per}[\![\delta]\!]$ is defined as $w\delta^{\varsigma}$ where $w \in \mathcal{E}_{per}$. In Cottenceau et al. (2014) it is shown that a monomial can be represented as

$$w\delta^{\varsigma} = (\bigoplus_{k=1}^{K} \gamma^{n_k}\mu_m\beta_b\gamma^{n'_k})\delta^{\varsigma}. \quad (13)$$

A polynomial in $\mathcal{E}^*_{per}[\![\delta]\!]$ is a sum of monomials $p = \bigoplus_{i=1}^{I} w_i\delta^{\varsigma_i}$ such that $\Gamma(w_i) = \Gamma(w_j)\ \forall i,j \in \{1, \cdots, I\}$. The gain $\Gamma(p)$ of a polynomial is defined to the gain of its coefficient, i.e.: $\Gamma(p) = \Gamma(w_i)$. A series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$ is said to be ultimately periodic if it can be written as $s = p \oplus q(\gamma^{\nu}\delta^{\tau})^*$, where $\nu, \tau \in \mathbb{N}$ and p, q are polynomials in $\mathcal{E}^*_{per}[\![\delta]\!]$ such that $\Gamma(p) = \Gamma(q)$. The gain of s is then defined to $\Gamma(s) = \Gamma(p)$.

Proposition 4. (Cottenceau et al. (2014)). Let s_1, s_2 be two ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$ then:

- $\Gamma(s_1) = \Gamma(s_2) \Rightarrow s_1 \oplus s_2$ is an ultimately periodic series, with $\Gamma(s_1 \oplus s_2) = \Gamma(s_1)$.
- $s_1 \otimes s_2$ (resp. $s_2 \otimes s_1$) is an ultimately periodic series, with $\Gamma(s_1 \otimes s_2) = \Gamma(s_1) \times \Gamma(s_2)$.
- $\Gamma(s_1) = 1 \Rightarrow s_1^*$ is an ultimately periodic series, with $\Gamma(s_1^*) = 1$.

4. CORE REPRESENTATION OF A SERIES IN $\mathcal{E}^*_{PER}[\![\delta]\!]$

In this section we show how a series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$ can be decomposed into a $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ matrix (core) combined with an input column vector and an output row vector. Thanks to this representation all operations between series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$ can be equally described by operations between their corresponding $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ core matrices.

4.1 Matrix Notations

In the following we define some matrices frequently used throughout this paper. In the core representation the input interface is a column vector defined as

$$\boldsymbol{\beta}_b := \begin{bmatrix}\beta_b\gamma^{b-1} & \cdots & \beta_b\gamma^1 & \beta_b\end{bmatrix}^T.$$

In analogy we define an output interface as a row vector

$$\boldsymbol{\mu}_m := \begin{bmatrix}\mu_m & \gamma^1\mu_m & \cdots & \gamma^{m-1}\mu_m\end{bmatrix}.$$

The index b (resp. m) determines the division (resp. multiplication) coefficient and gives the dimension of the vector. The identity matrix, denoted by $\boldsymbol{I}$, and the zero matrix, denoted by $\boldsymbol{\varepsilon}$, are given by

$$\boldsymbol{I} := \begin{bmatrix} \mathrm{e} & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon \\ \varepsilon & \cdots & \varepsilon & \mathrm{e} \end{bmatrix}, \quad \boldsymbol{\varepsilon} := \begin{bmatrix} \varepsilon & \cdots & \varepsilon \\ \vdots & & \vdots \\ \varepsilon & \cdots & \varepsilon \end{bmatrix}.$$

Finally, we define a particular square matrix,

$$\boldsymbol{E} := \begin{bmatrix} \mathrm{e} & \gamma^1 & \cdots & \gamma^1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \gamma^1 \\ \mathrm{e} & \cdots & \cdots & \mathrm{e} \end{bmatrix},$$

which is composed of operators $\{\mathrm{e}, \gamma^1\}$. If necessary, the dimension of $\boldsymbol{E}$ is stated as an index, e.g., $\boldsymbol{E}_m \in \{\mathrm{e}, \gamma^1\}^{m \times m}$.

4.2 Core-Form of a Series in $\mathcal{E}^*_{per}[\![\delta]\!]$

The decomposition of a series in $\mathcal{E}^*_{per}[\![\delta]\!]$ is carried out according to the following equation

$$s = \boldsymbol{\mu}_m \boldsymbol{X} \boldsymbol{\beta}_b. \quad (14)$$

This equation is called core-equation. Recall that $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ is a subdioid of $\mathcal{E}^*[\![\delta]\!]$. We say $\boldsymbol{Q} \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]^{m \times b}$ is a core of $s \in \mathcal{E}^*_{per}[\![\delta]\!]$, if $\boldsymbol{Q}$ is a solution of (14), i.e., $s = \boldsymbol{\mu}_m\boldsymbol{Q}\boldsymbol{\beta}_b$. Note that the dimension of the core matrix $\boldsymbol{Q} \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]^{m \times b}$ is determined by the periodicity (m, b) of the series s. In general, there exist several cores $\boldsymbol{Q}$ which solve (14). In the following, we show how to obtain a core $\boldsymbol{Q}$ for a given series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$. Furthermore, we prove that s admits a unique greatest core, denoted $\hat{\boldsymbol{Q}} \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]^{m \times b}$ (greatest with respect to the order relation in the dioid $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$, i.e., $\hat{\boldsymbol{Q}} \succeq \boldsymbol{Q}$).

Due to (13) and by using (7) $\gamma^n\mu_m = \gamma^{n-\lfloor n/m \rfloor m}\mu_m\gamma^{\lfloor n/m \rfloor}$ (resp. $\beta_b\gamma^{n'} = \gamma^{\lfloor n'/b \rfloor}\beta_b\gamma^{n'-\lfloor n'/b \rfloor b}$), a series $s = p \oplus q(\gamma^{\nu}\delta^{\tau})^* \in \mathcal{E}^*_{per}[\![\delta]\!]$ can be expressed as

$$s = \bigoplus_{i=1}^{I} \gamma^{n_i}\mu_m \gamma^{\bar{n}_i}\delta^{\varsigma_i}\beta_b\gamma^{n'_i} \oplus \bigoplus_{j=1}^{J}\gamma^{N_j}\mu_m\gamma^{\bar{N}_j}\delta^{\bar{\varsigma}_j}(\gamma^\nu\delta^\tau)^*\beta_b\gamma^{N'_j},$$
$$= \bigoplus_{i=1}^{I}\gamma^{n_i}\mu_m M_i\beta_b\gamma^{n'_i} \oplus \bigoplus_{j=1}^{J}\gamma^{N_j}\mu_m S_j\beta_b\gamma^{N'_j},$$

where the M_i are monomials and S_j are series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Furthermore, $0 \le n'_i, N'_j < b$ and $0 \le n_i, N_j < m$. In this form the entries of the $\boldsymbol{\beta}_b$-vector are separated on the right and the entries of the $\boldsymbol{\mu}_m$-vector are separated on the left of monomial M_i (resp. series S_j). Note that in general the "growing-term" $(\gamma^\nu\delta^\tau)^*$ of a series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$ cannot commute with the β_b operator. To bring the "growing-term" $(\gamma^\nu\delta^\tau)^*$ of a series to the left hand side of the β_b operator, ν must be a multiple of b, see (7). However, any series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$ can be rewritten such that the "growing-term" can commute with β_b by extending $(\gamma^\nu\delta^\tau)^*$ such that, $l\nu = lcm(\nu, b)$

$$(\gamma^\nu\delta^\tau)^* = (e \oplus \gamma^\nu\delta^\tau \oplus \cdots \oplus \gamma^{(l-1)\nu}\delta^{(l-1)\tau})(\gamma^{l\nu}\delta^{l\tau})^*.$$

Example 1. Consider the following series $s_e = \gamma^1\mu_3\beta_2\gamma^1\delta^2 \oplus (\gamma^3\mu_3\beta_2\gamma^1 \oplus \gamma^5\mu_3\beta_2)\delta^3(\gamma^1\delta^1)^*$. We first extend the "growing-term" $(\gamma^1\delta^1)^* = (\mathrm{e} \oplus \gamma^1\delta^1)(\gamma^2\delta^2)^*$. This leads to

$$s_e = \gamma^1\mu_3\beta_2\gamma^1\delta^2 \oplus ((\gamma^3\mu_3\beta_2\gamma^1 \oplus \gamma^5\mu_3\beta_2)\delta^3 \oplus (\gamma^6\mu_3\beta_2 \oplus \gamma^5\mu_3\beta_2\gamma^1)\delta^4)(\gamma^2\delta^2)^*.$$

In the next step we separate entries of $\boldsymbol{\mu}_m$ on the right and entries of $\boldsymbol{\beta}_b$ on the left for each monomial (resp. series).

$$s_e = \gamma^1\mu_3\underbrace{\delta^2}_{M_1}\beta_2\gamma^1 \oplus \mu_3\underbrace{\left(\gamma^2\delta^4(\gamma^1\delta^2)^*\right)}_{S_1}\beta_2 \oplus \mu_3\underbrace{\left(\gamma^1\delta^3(\gamma^1\delta^2)^*\right)}_{S_2}\beta_2\gamma^1$$
$$\oplus\, \gamma^2\mu_3\underbrace{\left(\gamma^1\delta^3(\gamma^1\delta^2)^*\right)}_{S_3}\beta_2 \oplus \gamma^2\mu_3\underbrace{\left(\gamma^1\delta^4(\gamma^1\delta^2)^*\right)}_{S_4}\beta_2\gamma^1$$

◁

For a given s we denote the set of monomials by $\mathscr{M} = \{M_1, \cdots, M_I\}$ and the set of series by $\mathscr{S} = \{S_1, \cdots, S_J\}$. Furthermore, the subsets $\mathscr{M}_{lg}$ (resp. $\mathscr{S}_{lg}$) are defined as

$$\forall l \in \{0, \cdots, m-1\},\ \forall g \in \{0, \cdots, b-1\},$$
$$\mathscr{M}_{lg} := \{M_i \in \mathscr{M} \mid \gamma^l\mu_m M_i\beta_b\gamma^g \in \bigoplus_{i=1}^{I}\gamma^{n_i}\mu_m M_i\beta_b\gamma^{n'_i}\},$$
$$\mathscr{S}_{lg} := \{S_j \in \mathscr{S} \mid \gamma^l\mu_m S_j\beta_b\gamma^g \in \bigoplus_{j=1}^{J}\gamma^{N_j}\mu_m S_j\beta_b\gamma^{N'_j}\}.$$

Example 2. For s_e of Example 1 we obtain the following subsets

$$\mathscr{M}_{11} = \{\delta^2\},\quad \mathscr{M}_{00} = \mathscr{M}_{01} = \mathscr{M}_{10} = \mathscr{M}_{20} = \mathscr{M}_{21} = \{\varepsilon\},$$
$$\mathscr{S}_{00} = \{\gamma^2\delta^4(\gamma^1\delta^2)^*\},\qquad \mathscr{S}_{01} = \{\gamma^1\delta^3(\gamma^1\delta^2)^*\},$$
$$\mathscr{S}_{20} = \{\gamma^1\delta^3(\gamma^1\delta^2)^*\},\qquad \mathscr{S}_{21} = \{\gamma^1\delta^4(\gamma^1\delta^2)^*\},$$
$$\mathscr{S}_{10} = \mathscr{S}_{11} = \{\varepsilon\}.$$

◁

The element $(\boldsymbol{Q})_{(l+1)(b-g)}$ of the core matrix is then obtained by

$$(\boldsymbol{Q})_{(l+1)(b-g)} = \bigoplus_{M \in \mathscr{M}_{lg}} M \oplus \bigoplus_{S \in \mathscr{S}_{lg}} S.$$

Hence, a series s can be expressed by $s = \boldsymbol{\mu}_m\boldsymbol{Q}\boldsymbol{\beta}_b$.

Example 3. The core-form of the series s_e considered in Example 1 is given by $\boldsymbol{\mu}_3\boldsymbol{Q}\boldsymbol{\beta}_2$ where

$$\boldsymbol{Q} = \begin{bmatrix} \gamma^1\delta^3(\gamma^1\delta^2)^* & \gamma^2\delta^4(\gamma^1\delta^2)^* \\ \delta^2 & \varepsilon \\ \gamma^1\delta^4(\gamma^1\delta^2)^* & \gamma^1\delta^3(\gamma^1\delta^2)^* \end{bmatrix}.$$

◁

Consider an input interface $\boldsymbol{\beta}_i$ and an output interface $\boldsymbol{\mu}_i$ of the same size. Then, the scalar product $\boldsymbol{\mu}_i\boldsymbol{\beta}_i$ is the identity e, since (9),

$$\boldsymbol{\mu}_i \otimes \boldsymbol{\beta}_i = \mu_i\beta_i\gamma^{i-1} \oplus \gamma^1\mu_i\beta_i\gamma^{i-2} \oplus \cdots \oplus \gamma^{i-1}\mu_i\beta_i = \mathrm{e}\,. \tag{15}$$

The dyadic product $\boldsymbol{\beta}_i \otimes \boldsymbol{\mu}_i$ is the $\boldsymbol{E}$ matrix of size $i \times i$,

$$\boldsymbol{\beta}_i \otimes \boldsymbol{\mu}_i, \tag{16}$$
$$= \begin{bmatrix} \beta_i\gamma^{i-1}\mu_i & \gamma^1\beta_i\mu_i & \gamma^1\beta_i\gamma^1\mu_i & \cdots & \gamma^1\beta_i\gamma^{i-2}\mu_i \\ \beta_i\gamma^{i-2}\mu_i & \beta_i\gamma^{i-1}\mu_i & \gamma^1\beta_i\mu_i & \cdots & \gamma^1\beta_i\gamma^{i-3}\mu_i \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_i\gamma^1\mu_i & \beta_i\gamma^2\mu_i & \beta_i\gamma^3\mu_i & \cdots & \gamma^1\beta_i\mu_i \\ \beta_i\mu_i & \beta_i\gamma^1\mu_i & \beta_i\gamma^2\mu_i & \cdots & \beta_i\gamma^{i-1}\mu_i \end{bmatrix},$$

since $\beta_i\gamma^n\mu_i = \mathrm{e}$ for $0 \le n < i$,

$$= \begin{bmatrix} \mathrm{e} & \gamma^1 & \cdots & \gamma^1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \gamma^1 \\ \mathrm{e} & \cdots & \cdots & \mathrm{e} \end{bmatrix} = \boldsymbol{E}.$$

Proposition 5. For the $\boldsymbol{E}$ matrix the following relations hold $\boldsymbol{E} \otimes \boldsymbol{E} = \boldsymbol{E}$; $\boldsymbol{E} \otimes \boldsymbol{\beta}_i = \boldsymbol{\beta}_i$; $\boldsymbol{\mu}_i \otimes \boldsymbol{E} = \boldsymbol{\mu}_i$.

Proof.

$$\boldsymbol{E} \otimes \boldsymbol{E} = \boldsymbol{\beta}_i \otimes \boldsymbol{\mu}_i \otimes \boldsymbol{\beta}_i \otimes \boldsymbol{\mu}_i = \boldsymbol{\beta}_i \otimes \mathrm{e} \otimes \boldsymbol{\mu}_i = \boldsymbol{E},$$
$$\boldsymbol{E} \otimes \boldsymbol{\beta}_i = \boldsymbol{\beta}_i \otimes \boldsymbol{\mu}_i \otimes \boldsymbol{\beta}_i = \boldsymbol{\beta}_i \otimes \mathrm{e} = \boldsymbol{\beta}_i,$$
$$\boldsymbol{\mu}_i \otimes \boldsymbol{E} = \boldsymbol{\mu}_i \otimes \boldsymbol{\beta}_i \otimes \boldsymbol{\mu}_i = \mathrm{e} \otimes \boldsymbol{\mu}_i = \boldsymbol{\mu}_i.$$

□

Proposition 6. For $\boldsymbol{D} \in \mathcal{E}^*[\![\delta]\!]^{1\times n}$ and $\boldsymbol{P} \in \mathcal{E}^*[\![\delta]\!]^{n\times 1}$, we have

$$\boldsymbol{\mu}_i \circ\!\!\!\backslash \boldsymbol{D} = \boldsymbol{\beta}_i \otimes \boldsymbol{D}, \qquad \boldsymbol{P} \phi\!\!\!/ \boldsymbol{\beta}_i = \boldsymbol{P} \otimes \boldsymbol{\mu}_i. \tag{17}$$

For $\boldsymbol{M} \in \mathcal{E}^*[\![\delta]\!]^{n\times m}$, $\boldsymbol{N} \in \mathcal{E}^*[\![\delta]\!]^{b\times n}$, $\tilde{\boldsymbol{M}} = \boldsymbol{M}\boldsymbol{E}$ and $\tilde{\boldsymbol{N}} = \boldsymbol{E}\boldsymbol{N}$, we have

$$\tilde{\boldsymbol{M}} \phi\!\!\!/ \boldsymbol{\mu}_i = \tilde{\boldsymbol{M}} \otimes \boldsymbol{\beta}_i, \qquad \boldsymbol{\beta}_i \circ\!\!\!\backslash \tilde{\boldsymbol{N}} = \boldsymbol{\mu}_i \otimes \tilde{\boldsymbol{N}}. \tag{18}$$

Proof. See Appendix B. □

Proposition 7. Let $s = \boldsymbol{\mu}_m\boldsymbol{Q}\boldsymbol{\beta}_b \in \mathcal{E}^*_{per}[\![\delta]\!]$, the core equation $s = \boldsymbol{\mu}_m\boldsymbol{X}\boldsymbol{\beta}_b$ has a unique greatest solution, denoted $\hat{\boldsymbol{Q}}$ and given by

$$\hat{\boldsymbol{Q}} = \boldsymbol{E}_m\boldsymbol{Q}\boldsymbol{E}_b. \tag{19}$$

Proof. Consider the inequality $\boldsymbol{\mu}_m\tilde{\boldsymbol{X}}\boldsymbol{\beta}_b \preceq \boldsymbol{\mu}_m\boldsymbol{Q}\boldsymbol{\beta}_b = s$. The greatest solution for $\tilde{\boldsymbol{X}}$ is

$$\boldsymbol{\mu}_m \circ\!\!\!\backslash \boldsymbol{\mu}_m\boldsymbol{Q}\boldsymbol{\beta}_b \phi\!\!\!/ \boldsymbol{\beta}_b = \boldsymbol{\beta}_m\boldsymbol{\mu}_m\boldsymbol{Q}\boldsymbol{\beta}_b\boldsymbol{\mu}_b = \boldsymbol{E}_m\boldsymbol{Q}\boldsymbol{E}_b = \hat{\boldsymbol{Q}}.$$

Furthermore, we can check that $\hat{\boldsymbol{Q}}$ solves the core equation. Since $\boldsymbol{\mu}_m = \boldsymbol{\mu}_m\boldsymbol{E}_m$ and $\boldsymbol{\beta}_b = \boldsymbol{E}_b\boldsymbol{\beta}_b$,

$$\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b = \boldsymbol{\mu}_m\boldsymbol{E}_m\boldsymbol{Q}\boldsymbol{E}_b\boldsymbol{\beta}_b = \boldsymbol{\mu}_m\boldsymbol{Q}\boldsymbol{\beta}_b = s.$$

□

Remark 1. The greatest core matrix $\hat{\boldsymbol{Q}}$ has the following properties. Since: $\boldsymbol{E} \otimes \boldsymbol{E} = \boldsymbol{E}$, $\boldsymbol{E}\hat{\boldsymbol{Q}} = \boldsymbol{EEQE} = \hat{\boldsymbol{Q}}$; $\hat{\boldsymbol{Q}}\boldsymbol{E} = \boldsymbol{EQEE} = \hat{\boldsymbol{Q}}$. ◁

Example 4. The greatest core of the series considered in Example 3 is given by

$$\hat{\boldsymbol{Q}} = \boldsymbol{E}_3\boldsymbol{Q}\boldsymbol{E}_2,$$
$$= \begin{bmatrix} e & \gamma^1 & \gamma^1 \\ e & e & \gamma^1 \\ e & e & e \end{bmatrix} \begin{bmatrix} \gamma^1\delta^3(\gamma^1\delta^2)^* & \gamma^2\delta^4(\gamma^1\delta^2)^* \\ \delta^2 & \varepsilon \\ \gamma^1\delta^4(\gamma^1\delta^2)^* & \gamma^1\delta^3(\gamma^1\delta^2)^* \end{bmatrix} \begin{bmatrix} e & \gamma^1 \\ e & e \end{bmatrix},$$
$$= \begin{bmatrix} \gamma^1\delta^3(\gamma^1\delta^2)^* & \gamma^2\delta^4(\gamma^1\delta^2)^* \\ \delta^2 \oplus \gamma^1\delta^3(\gamma^1\delta^2)^* & \gamma^1\delta^2(\gamma^1\delta^2)^* \\ \delta^2(\gamma^1\delta^2)^* & \gamma^1\delta^3(\gamma^1\delta^2)^* \end{bmatrix}.$$
◁

Similarly to the extension of the basic $\mu_m\beta_b$ operator in (9) we can extend matrix $\hat{\boldsymbol{Q}}$ in the core-form of a series.

Proposition 8. A series $s = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b \in \mathcal{E}^*_{per}[\![\delta]\!]$ can be expressed with a multiple period (nm, nb) by extending the core matrix $\hat{\boldsymbol{Q}}$, i.e., $s = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b = \boldsymbol{\mu}_{nm}\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{nb}$, where $\hat{\boldsymbol{Q}}' \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]^{nm\times nb}$ and is given by

$$\hat{\boldsymbol{Q}}' = \begin{bmatrix} \beta_n\gamma^{n-1}\hat{\boldsymbol{Q}}\mu_n & \cdots & \beta_n\gamma^{n-1}\hat{\boldsymbol{Q}}\gamma^{n-1}\mu_n \\ \vdots & & \vdots \\ \beta_n\hat{\boldsymbol{Q}}\mu_n & \cdots & \beta_n\hat{\boldsymbol{Q}}\gamma^{n-1}\mu_n \end{bmatrix}.$$

Proof. See Appendix B. □

4.3 Operations between Core Matrices

Because of Prop. 8, by extending the core-form if necessary, two series with equal gain can be expressed with their least common period.

Proposition 9. (Sum of series).
Let $s = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b$, $s' = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_b \in \mathcal{E}^*_{per}[\![\delta]\!]$, the sum $s \oplus s' = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}''\boldsymbol{\beta}_b$, where $\hat{\boldsymbol{Q}}'' = (\hat{\boldsymbol{Q}} \oplus \hat{\boldsymbol{Q}}')$ is again a greatest core.

Proof.
$$\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b \oplus \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_b = \boldsymbol{\mu}_m(\boldsymbol{EQE} \oplus \boldsymbol{EQ'E})\boldsymbol{\beta}_b$$
$$= \boldsymbol{\mu}_m \underbrace{\boldsymbol{E}(\boldsymbol{Q} \oplus \boldsymbol{Q}')\boldsymbol{E}}_{\hat{\boldsymbol{Q}}''}\boldsymbol{\beta}_b$$
□

Proposition 10. (Product of series).
Let $s = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b \in \mathcal{E}^*_{per}[\![\delta]\!]$ and $s' = \boldsymbol{\mu}_b\hat{\boldsymbol{Q}}\boldsymbol{\beta}_{b'} \in \mathcal{E}^*_{per}[\![\delta]\!]$, the product $s \otimes s' = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}''\boldsymbol{\beta}_{b'}$, where $\hat{\boldsymbol{Q}}'' = \hat{\boldsymbol{Q}}\hat{\boldsymbol{Q}}'$ is again a greatest core.

Proof.
$$\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b\boldsymbol{\mu}_b\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{b'} = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{E}\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{b'} = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_b,$$
Furthermore: $\hat{\boldsymbol{Q}}\hat{\boldsymbol{Q}}' = \boldsymbol{EQEEQ'E} = \hat{\boldsymbol{Q}}''$.
□

Proposition 11. (Kleene star of series). Let $s = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_m \in \mathcal{E}^*_{per}[\![\delta]\!]$. Then, s^* can be obtained by

$$s^* = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}^*\boldsymbol{\beta}_m. \tag{20}$$

Proof. In this case, $\Gamma(s) = 1$ means that $\hat{\boldsymbol{Q}}$ is a square matrix.
$$s^* = e \oplus \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_m \oplus \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_m\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_m \oplus \cdots$$
Since: $e = \boldsymbol{\mu}_m\boldsymbol{\beta}_m$ (15), $\boldsymbol{E} = \boldsymbol{\beta}_m\boldsymbol{\mu}_m$ (16) and $\boldsymbol{E}\hat{\boldsymbol{Q}} = \hat{\boldsymbol{Q}}$ (Remark 1),
$$= \boldsymbol{\mu}_m\boldsymbol{\beta}_m \oplus \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_m \oplus \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}^2\boldsymbol{\beta}_m \oplus \cdots$$
$$= \boldsymbol{\mu}_m(\boldsymbol{I} \oplus \hat{\boldsymbol{Q}} \oplus \hat{\boldsymbol{Q}}^2 \oplus \cdots)\boldsymbol{\beta}_m$$
$$= \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}^*\boldsymbol{\beta}_m$$
□

4.4 Comparison of Kleene Star Algorithms

In Cottenceau et al. (2014) it is shown that the Kleene star of a polynomial, $p = \bigoplus_{i=1}^{I} \gamma^{n_i}\mu_m\beta_m\gamma^{n'_i}\delta^{\varsigma_i} \in \mathcal{E}^*_{per}[\![\delta]\!]$, can be obtained by the following computation

$$p^* = e \oplus \left(\bigoplus_{i=1}^{I}\bigoplus_{j=1}^{I}(\boldsymbol{\psi}^*)_{ij}\gamma^{n_i}\mu_m\beta_m\gamma^{n'_j}\delta^{\varsigma_j}\right),$$

where $\boldsymbol{\psi} \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]^{I\times I}$ is a matrix such that $\forall a, b \in \{1,\ldots,I\}$, $(\boldsymbol{\psi})_{ab} = \gamma^{\lfloor (n'_a+n_b)/m\rfloor m}\delta^{\varsigma_a}$. For large I this algorithm is computationally very expensive since it scales with the number I of monomials in the polynomial. First, we have to calculate the star of $\boldsymbol{\psi} \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]^{I\times I}$. Second, we have the sum of I^2 ultimately periodic series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$. The approach presented in Prop. 11 is in general computationally less expensive, since we only have to calculate the star of the core matrix $\hat{\boldsymbol{Q}}$ which is of size $m \times m$ (determined by the period). Roughly speaking, if the number of monomials in the polynomial is bigger than the period of the polynomial, then the algorithm suggested in Prop. 11 is preferable.

4.5 Division in $\mathcal{E}^*_{per}[\![\delta]\!]$

Proposition 12. Let $s = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b$, $s' = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{b'}$ be ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. Then,

$$s' \backslash\!\!\!\circ\, s = \boldsymbol{\mu}_{b'}(\hat{\boldsymbol{Q}}' \backslash\!\!\!\circ\, \hat{\boldsymbol{Q}})\boldsymbol{\beta}_b,$$

Proof. since (A.1) and $\boldsymbol{\mu}_m \backslash\!\!\!\circ\,(\boldsymbol{\mu}_m\hat{\boldsymbol{Q}},) = \boldsymbol{\beta}_m\boldsymbol{\mu}_m\hat{\boldsymbol{Q}} = \hat{\boldsymbol{Q}}$,
$$\left(\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{b'}\right) \backslash\!\!\!\circ \left(\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b\right) = \left(\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{b'}\right) \backslash\!\!\!\circ \left(\boldsymbol{\mu}_m \backslash\!\!\!\circ (\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b)\right),$$
$$= \left(\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{b'}\right) \backslash\!\!\!\circ \left(\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b\right).$$
Since $\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b = \hat{\boldsymbol{Q}} /\!\!\!\circ\, \boldsymbol{\mu}_b$ see Prop. 6, (A.1) and (A.2),
$$= \left(\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_{b'}\right) \backslash\!\!\!\circ \left(\hat{\boldsymbol{Q}} /\!\!\!\circ\, \boldsymbol{\mu}_b\right) = \boldsymbol{\beta}_{b'} \backslash\!\!\!\circ \left(\hat{\boldsymbol{Q}}' \backslash\!\!\!\circ (\hat{\boldsymbol{Q}} /\!\!\!\circ\, \boldsymbol{\mu}_b)\right)$$
$$= \boldsymbol{\beta}_{b'} \backslash\!\!\!\circ \left((\hat{\boldsymbol{Q}}' \backslash\!\!\!\circ\, \hat{\boldsymbol{Q}}) /\!\!\!\circ\, \boldsymbol{\mu}_b\right) = \boldsymbol{\mu}_{b'}(\hat{\boldsymbol{Q}}' \backslash\!\!\!\circ\, \hat{\boldsymbol{Q}})\boldsymbol{\beta}_b.$$
□

Proposition 13. Let $s = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b$, $s' = \boldsymbol{\mu}_{m'}\hat{\boldsymbol{Q}}'\boldsymbol{\beta}_b$ be ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. Then,

$$s /\!\!\!\circ\, s' = \boldsymbol{\mu}_m(\hat{\boldsymbol{Q}} /\!\!\!\circ\, \hat{\boldsymbol{Q}}')\boldsymbol{\beta}_{m'}.$$

Proof. The proof is analogous to the proof of Prop. 12. □

Additionally, it can be shown that $\hat{\boldsymbol{Q}}' \backslash\!\!\!\circ\, \hat{\boldsymbol{Q}}$ and $\hat{\boldsymbol{Q}}' /\!\!\!\circ\, \hat{\boldsymbol{Q}}$ are greatest cores.

5. MODELING AND CONTROL OF SISO WBTEGs

5.1 Transfer Matrices of WBTEGs

From now on we assume that WBTEGs are operating under the earliest functioning rule. Then the firing relation between two transitions, composing a basic directed path $t_j \rightarrow p_i \rightarrow t_o$, can be described by E-operators combined with the time-shift operator δ^{ς} as follows, $x_o = \beta_{w(p_i,t_o)}\delta^{(\boldsymbol{\phi})_i}\gamma^{(\boldsymbol{M}_0)_i}\mu_{w(t_j,p_i)}x_j$, where x_j (resp. x_o) refers to the counter function of transition t_j (resp. t_o), $w(p_i, t_o)$ (resp. $w(t_j, p_i)$) is the weight of the arc (p_i, t_o) (resp.(t_j, p_i)), $(\boldsymbol{\phi})_i$ is the holding time of place p_i and $(\boldsymbol{M}_0)_i$ is the initial marking of p_i. For instance, take the path $t_2 \rightarrow p_3 \rightarrow t_3$ of the WBTEG in Fig. 2, the firing relation between t_2 and t_3 corresponds to an operator representation $x_3 = \beta_2\delta^1\gamma^1\mu_1 x_2$. Therefore, the firing relation between internal, input and output transition in a WBTEG can be described by a state space representation:

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x},$$

where $\boldsymbol{x}$ (resp. $\boldsymbol{u}, \boldsymbol{y}$) refers to counter functions of internal (resp. input, output) transitions and $\boldsymbol{A} \in \mathcal{E}^*_{per}[\![\delta]\!]^{n\times n}$, $\boldsymbol{B} \in \mathcal{E}^*_{per}[\![\delta]\!]^{n\times m}$, $\boldsymbol{C} \in \mathcal{E}^*_{per}[\![\delta]\!]^{p\times n}$.

Theorem 3. (Cottenceau et al. (2014)). For an m-inputs and p-outputs WBTEG, the entries of the transfer matrix $\boldsymbol{H} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}$ are ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. ◁

Example 5. Consider the WBTEG in Fig. 2. The transfer function h_s describes the firing relation between input transition t_1 and output transition t_3 and is given by

$$h_s = \beta_2\gamma^1\delta^1(\gamma^3\delta^1)^*\mu_3 = (\mu_3\beta_2\gamma^1 \oplus \gamma^2\mu_3\beta_2)\delta^1(\gamma^1\delta^1)^*,$$
$$= \boldsymbol{\mu}_3\hat{\boldsymbol{Q}}\boldsymbol{\beta}_2 = \boldsymbol{\mu}_3 \begin{bmatrix} \delta^1(\gamma^1\delta^2)^* & \gamma^1\delta^2(\gamma^1\delta^2)^* \\ \delta^1(\gamma^1\delta^2)^* & \gamma^1\delta^2(\gamma^1\delta^2)^* \\ \delta^2(\gamma^1\delta^2)^* & \delta^1(\gamma^1\delta^2)^* \end{bmatrix} \boldsymbol{\beta}_2.$$

◁

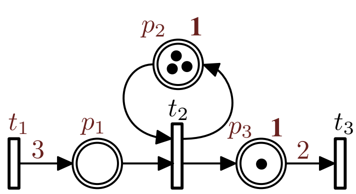

Fig. 2. A SISO WBTEG.

5.2 Dynamic feedback synthesis for SISO WBTEGs

Model reference control is a method to manipulate the dynamics of a plant (WBTEG) such that a given reference model is matched (Maia et al. (2003)). For the class of SISO WBTEGs such a reference model can be specified by a transfer function $g \in \mathcal{E}^*_{per}[\![\delta]\!]$. The control problem is then to find an output feedback f for a plant model $h \in \mathcal{E}^*_{per}[\![\delta]\!]$ such that the closed loop system $(hf)^*h$ satisfies $(hf)^*h \preceq g$. Moreover, we are looking for the greatest possible feedback f. This inequality is not always solvable. However, in the particular case where the reference model g is equal to the transfer function h of the plant, an optimal feedback f can be obtained by applying residuation theory, $f = h \circ\!\!\!\backslash h \circ\!\!\!/ h$, Maia et al. (2003). This is often referred to as "just-in-time" control as the controller delays all input events as much as possible without slowing done the plant, i.e., it is said neutral for this reason. Moreover, the feedback stabilizes the system by enforcing the closed loop system to be strongly connected. Consider a SISO WBTEG with a transfer function $h = \boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b$, then because of Prop. 12 and Prop. 13, the greatest neutral output feedback is given by

$$f = h \circ\!\!\!\backslash h \circ\!\!\!/ h = \left((\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b) \circ\!\!\!\backslash (\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b)\right) \circ\!\!\!/ \left(\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b\right),$$
$$= \left(\boldsymbol{\mu}_b\left(\hat{\boldsymbol{Q}} \circ\!\!\!\backslash \hat{\boldsymbol{Q}}\right)\boldsymbol{\beta}_b\right) \circ\!\!\!/ \left(\boldsymbol{\mu}_m\hat{\boldsymbol{Q}}\boldsymbol{\beta}_b\right) = \boldsymbol{\mu}_b\left(\left(\hat{\boldsymbol{Q}} \circ\!\!\!\backslash \hat{\boldsymbol{Q}}\right) \circ\!\!\!/ \hat{\boldsymbol{Q}}\right)\boldsymbol{\beta}_m.$$

Since, $\hat{\boldsymbol{Q}} \in \mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]^{m\times m}$ is a matrix in $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ we can use the existing tools introduced in Maia et al. (2003) for the feedback design process of WBTEGs.

Example 6. Consider the WBTEG in Fig. 2, the greatest neutral output feedback is

$$f = \boldsymbol{\mu}_2(\hat{\boldsymbol{Q}} \circ\!\!\!\backslash \hat{\boldsymbol{Q}} \circ\!\!\!/ \hat{\boldsymbol{Q}})\boldsymbol{\beta}_3,$$
$$= \boldsymbol{\mu}_2 \begin{bmatrix} \gamma^1\delta^1(\gamma^1\delta^2)^* & \gamma^1\delta^1(\gamma^1\delta^2)^* & \gamma^1(\gamma^1\delta^2)^* \\ (\gamma^1\delta^2)^* & (\gamma^1\delta^2)^* & \gamma^1\delta^1(\gamma^1\delta^2)^* \end{bmatrix} \boldsymbol{\beta}_3.$$

We can express this feedback f as an equivalent series in $\mathcal{E}^*_{per}[\![\delta]\!]$,

$$f = (\gamma^1\delta^1)^*(\gamma^2\mu_2\beta_3 \oplus \gamma^1\mu_2\beta_3\gamma^1).$$

Fig. 3 illustrates the closed loop system. The feedback bounds the amount of tokens in place p_1, p_3 while the throughput of the system is preserved. The feedback of this example is computed with help of the software tools Hardouin et al. (2009). ◁

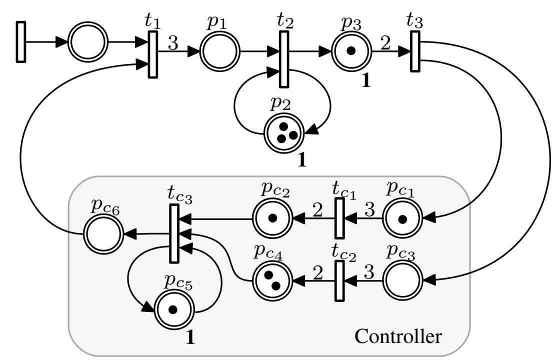

Fig. 3. Closed loop system.

6. CONCLUSION

In this work we present the algebraic tools to address the problem of "just-in-time" feedback synthesis for the class of SISO WBTEGs. The introduced method is based on a decomposition of the transfer function of a SISO WBTEG such that the dynamic of the system can be expressed in an event-invariant core matrix in $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$. An advantage of this method is that the already existing tools for feedback synthesis for conventional TEGs can be directly applied to SISO WBTEGs (a library for manipulating series in $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ is available in Hardouin et al. (2009)). The generalization of the problem to multiple-input and multiple-output (MIMO) WBTEGs will be investigated in further work.

Appendix A. FORMULAS OF RESIDUATION

$$(ab) \circ\!\!\!\backslash x = b \circ\!\!\!\backslash (a \circ\!\!\!\backslash x) \qquad x \circ\!\!\!/ (ba) = (x \circ\!\!\!/ a) \circ\!\!\!/ (b) \qquad \text{(A.1)}$$
$$(a \circ\!\!\!\backslash x) \circ\!\!\!/ b = a \circ\!\!\!\backslash (x \circ\!\!\!/ b) \qquad a \circ\!\!\!\backslash (x \circ\!\!\!/ b) = (a \circ\!\!\!\backslash x) \circ\!\!\!/ b \qquad \text{(A.2)}$$

Appendix B. PROOFS

Proof. of Prop. 6. (17) for the left product, by definition of the residual mapping $\boldsymbol{\mu}_m \circ\!\!\!\backslash \boldsymbol{D}$ is the greatest solution of the following inequality

$$\boldsymbol{\mu}_m \otimes \boldsymbol{X} \preceq \boldsymbol{D},$$
$$\boldsymbol{\mu}_m \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} \preceq [d_1 \cdots d_n].$$

This matrix inequality can be transformed into a set of n inequalities,

$$\mu_m x_{11} \oplus \gamma^1 \mu_m x_{21} \oplus \cdots \oplus \gamma^{m-1} \mu_m x_{m1} \preceq d_1,$$
$$\vdots$$
$$\mu_m x_{1n} \oplus \gamma^1 \mu_m x_{2n} \oplus \cdots \oplus \gamma^{m-1} \mu_m x_{mn} \preceq d_n.$$

Because of Prop. 3, for each inequality $i \in \{1, \cdots, n\}$ we obtain

$$x_{1i} \preceq \mu_m \diamond\!\!\!\backslash d_i = \beta_m \gamma^{m-1} d_i,$$
$$\vdots$$
$$x_{mi} \preceq (\gamma^{m-1}\mu_m) \diamond\!\!\!\backslash d_i = \beta_m d_i.$$

Rewriting the inequalities into matrix form leads to

$$\boldsymbol{X} \preceq \boldsymbol{\mu}_m \diamond\!\!\!\backslash \boldsymbol{D} = \begin{bmatrix} \beta_m \gamma^{m-1} \\ \vdots \\ \beta_m \end{bmatrix} \boldsymbol{D} = \boldsymbol{\beta}_m \otimes \boldsymbol{D}.$$

For the right product by $\boldsymbol{\beta}_b$ we have

$$\boldsymbol{X}\boldsymbol{\beta}_b \preceq \boldsymbol{P} \Leftrightarrow \boldsymbol{X} \preceq \boldsymbol{P} \diamond\!\!\!/ \boldsymbol{\beta}_b,$$

where $\boldsymbol{X}$ is of size $n \times m$ and $\boldsymbol{P}$ is of size $n \times 1$. We obtain for each $i \in \{1, \cdots, n\}$ the following inequalities

$$x_{i1} \preceq p_i \diamond\!\!\!/ (\beta_b \gamma^{b-1}) = p_i \mu_b,$$
$$\vdots$$
$$x_{in} \preceq p_i \diamond\!\!\!/ \beta_b = p_i \gamma^{b-1} \mu_b.$$

This can be expressed in matrix form

$$\boldsymbol{X} \preceq \boldsymbol{P} \diamond\!\!\!/ \boldsymbol{\beta}_b = \boldsymbol{P}\left[\mu_b \; \gamma^1 \mu_b \; \cdots \; \gamma^{b-1}\mu_b\right] = \boldsymbol{P} \otimes \boldsymbol{\mu}_b.$$

To prove (18), since $\boldsymbol{\beta}_i \boldsymbol{\mu}_i = \boldsymbol{E} = \boldsymbol{E}\boldsymbol{E}$ and due to (17) $\boldsymbol{P}\boldsymbol{\mu}_i = \boldsymbol{P} \diamond\!\!\!/ \boldsymbol{\beta}_i$ we can write

$$\tilde{\boldsymbol{M}} \diamond\!\!\!/ \boldsymbol{\mu}_i = (\tilde{\boldsymbol{M}}\boldsymbol{\beta}_i\boldsymbol{\mu}_i) \diamond\!\!\!/ \boldsymbol{\mu}_i = ((\tilde{\boldsymbol{M}}\boldsymbol{\beta}_i) \diamond\!\!\!/ \boldsymbol{\beta}_i) \diamond\!\!\!/ \boldsymbol{\mu}_i.$$

Since, $(x \diamond\!\!\!/ a) \diamond\!\!\!/ b = x \diamond\!\!\!/ (ba)$ (A.1) and $\boldsymbol{\mu}_i \boldsymbol{\beta}_i = \mathrm{e}$ (see 15),

$$((\tilde{\boldsymbol{M}}\boldsymbol{\beta}_i) \diamond\!\!\!/ \boldsymbol{\beta}_i) \diamond\!\!\!/ \boldsymbol{\mu}_i = (\tilde{\boldsymbol{M}}\boldsymbol{\beta}_i) \diamond\!\!\!/ (\boldsymbol{\mu}_i \boldsymbol{\beta}_i) = (\tilde{\boldsymbol{M}}\boldsymbol{\beta}_i) \diamond\!\!\!/ \mathrm{e} = \tilde{\boldsymbol{M}}\boldsymbol{\beta}_i.$$

The proof of $\boldsymbol{\beta}_i \diamond\!\!\!\backslash \tilde{\boldsymbol{N}} = \boldsymbol{\mu}_i \otimes \tilde{\boldsymbol{N}}$ is analogous. □

Proof. of Prop. 8. We can extend the core matrix $\boldsymbol{Q}$ of a series, i.e.,

$$s = \boldsymbol{\mu}_m \boldsymbol{Q} \boldsymbol{\beta}_b = \boldsymbol{\mu}_{nm} \underbrace{\boldsymbol{\beta}_{nm} \boldsymbol{\mu}_m \boldsymbol{Q} \boldsymbol{\beta}_b \boldsymbol{\mu}_{nb}}_{\hat{\boldsymbol{Q}}'} \boldsymbol{\beta}_{nb}.$$

Since, $\beta_{nm}\gamma^{mn-1} = \beta_n \beta_m \gamma^{m(n-1)} \gamma^{m-1} = \beta_n \gamma^{n-1} \beta_m \gamma^{m-1}$ then

$$\boldsymbol{\beta}_{nm} = \begin{bmatrix} \begin{bmatrix} \beta_n \gamma^{n-1} \beta_m \gamma^{m-1} \\ \vdots \\ \beta_n \gamma^{n-1} \beta_m \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \beta_n \beta_m \gamma^{m-1} \\ \vdots \\ \beta_n \beta_m \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \beta_n \gamma^{n-1} \boldsymbol{\beta}_m \\ \vdots \\ \beta_n \boldsymbol{\beta}_m \end{bmatrix}.$$

This leads to

$$\boldsymbol{\beta}_{nm} \boldsymbol{\mu}_m = \begin{bmatrix} \beta_n \gamma^{n-1} \boldsymbol{E} \\ \vdots \\ \beta_n \boldsymbol{E} \end{bmatrix}.$$

Respectively $\boldsymbol{\beta}_b \boldsymbol{\mu}_{nb}$ is given by

$$\boldsymbol{\beta}_b \boldsymbol{\mu}_{nb} = \left[\boldsymbol{E}\mu_n \; \cdots \; \boldsymbol{E}\gamma^{n-1}\mu_n\right].$$

Finally we obtain

$$\boldsymbol{Q}' = \begin{bmatrix} \beta_n \gamma^{n-1} \boldsymbol{E} \\ \vdots \\ \beta_n \boldsymbol{E} \end{bmatrix} \boldsymbol{Q} \left[\boldsymbol{E}\mu_n \; \cdots \; \boldsymbol{E}\gamma^{n-1}\mu_n\right],$$
$$= \begin{bmatrix} \beta_n \gamma^{n-1} \hat{\boldsymbol{Q}} \mu_n & \cdots & \beta_n \gamma^{n-1} \hat{\boldsymbol{Q}} \gamma^{n-1} \mu_n \\ \vdots & & \vdots \\ \beta_n \hat{\boldsymbol{Q}} \mu_n & \cdots & \beta_n \hat{\boldsymbol{Q}} \gamma^{n-1} \mu_n \end{bmatrix}.$$

The extended core is a matrix in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, since $\beta_n \gamma^\nu \mu_n = \gamma^{\lfloor \nu/n \rfloor n}$. Furthermore, the extended core $\boldsymbol{Q}'$ is a greatest core since

$$\hat{\boldsymbol{Q}}' = \boldsymbol{E}\boldsymbol{\beta}_{nm}\boldsymbol{\mu}_m \boldsymbol{Q} \boldsymbol{\beta}_b \boldsymbol{\mu}_{nb} \boldsymbol{E},$$
$$= \boldsymbol{\beta}_{nm} \underbrace{\boldsymbol{\mu}_{nm}\boldsymbol{\beta}_{nm}}_{\mathrm{e}} \boldsymbol{\mu}_m \boldsymbol{Q} \boldsymbol{\beta}_b \underbrace{\boldsymbol{\mu}_{nb}\boldsymbol{\beta}_{nb}}_{\mathrm{e}} \boldsymbol{\mu}_{nb},$$
$$= \boldsymbol{\beta}_{nm}\boldsymbol{\mu}_m \boldsymbol{Q} \boldsymbol{\beta}_b \boldsymbol{\mu}_{nb} = \boldsymbol{Q}'.$$

□

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