Model Decomposition of Weight-Balanced Timed Event Graphs in Dioids: Application to Control Synthesis

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Abstract: For Timed Event Graphs (TEGs), model reference control is a well known approach to stabilize and to improve the performance of a system. This method is based on dioid and residuation theory. In this work, we study model reference control for the class of single-input and single-output (SISO) Weight-Balanced Timed Event Graphs (WBTEGs), which is an extension of TEGs and exhibits event-variant behavior. By modeling the behavior of WBTEGs in a dioid structure, we propose a decomposition of the dynamic behavior into an event-variant and an event-invariant part. We further show that the event-variant part is invertible, hence the problem of model reference control for WBTEGs can be reduced to the case of conventional TEGs.

Keywords: Dioid, Weighted Timed Event Graph (WTEG), controller synthesis, discrete-event systems.

1. INTRODUCTION AND MOTIVATION

TEGs are a subclass of timed Petri nets where each place has exactly one input and one output transition and all arcs have weight 1. Weighted Timed Event Graphs (WTEGs) are an extension of TEGs where the weights on the arcs can take values in \( \mathbb{N} = \{1, 2, \cdots \} \). This formalism is popular to model systems ruled by synchronization, such as manufacturing processes or transport networks. The weights associated with arcs in WTEGs are suitable to express batch/split processes, for instance, when several occurrences of events are needed to induce a following event or when one event can result in several following events. This leads to an event-variant behavior which cannot be expressed by conventional TEGs. An equivalent graphical representation for WTEGs are Synchronous Data-Flow (SDF) graphs, and homogeneous SDF graphs are an equivalent representation of TEGs. WTEGs and SDF graphs have been widely studied e.g., for WTEGs see Marchetti and Munier-Kordon (2010), Teruel et al. (1992), and for SDF graphs see Sriram and Bhattacharyya (2000). SDF graphs are a popular framework for modeling and analyzing real-time embedded and multiprocessor systems. Most of the related work focuses on throughput analysis, which is a key property for the performance of those systems. For TEGs (resp. homogeneous SDF graphs), many tools have been developed for performance evaluation, for instance, their behavior can be modeled as linear state space representations in a tropical algebra structure called (max,+) algebra. In Baccelli et al. (1992) and Cochet-Terrasson et al. (1998), spectral analysis for linear (max,+) systems is introduced, which has an application in throughput calculation for TEGs. However, these methods cannot be directly applied to WTEGs. Munier (1993) suggests a transformation, which maps subclass of WTEGs into equivalent TEGs. A similar transformation is known for SDF graphs, see Sriram and Bhattacharyya (2000). However, the transformation can be computationally quite expensive, since the transformation significantly increases the number of transitions in the corresponding TEG.

Recently, Cottenceau et al. (2014) introduced a dioid, denoted \( \mathcal{E}^*[\delta] \), for modeling and analysis of an important subclass of WTEGs - the class of WTEGs where parallel paths have balanced weights. This class is therefore called Weight-Balanced Timed Event Graphs (WBTEGs). The dioid \( \mathcal{E}^*[\delta] \) is composed of four basic operators, event shift \( \gamma \), time shift \( \delta \), event multiplication (split) \( \mu \) and event division (batch) \( \beta \). \( \mathcal{E}^*[\delta] \) gives a natural way to model a WBTEG in a state space representation, similar to the modeling process of TEGs in \( \mathcal{M}^{|a|}_{n,\gamma,\delta} \), see Baccelli et al. (1992). Furthermore, it is shown that the input-output behavior of WBTEGs can be described by ultimately periodic series in \( \mathcal{E}^*[\delta] \). To obtain these transfer functions, the Kleene star operation for elements in \( \mathcal{E}^*[\delta] \) plays a key role. In this work, we show how elements in the dioid \( \mathcal{E}^*[\delta] \) can be decomposed into an event-variant part and an event-invariant core given by a matrix in \( \mathcal{M}^{|a|}_{n,\gamma,\delta} \). Furthermore, we show that all basic operations on the dioid \( \mathcal{E}^*[\delta] \), can be reduced to operations between matrices in \( \mathcal{M}^{|a|}_{n,\gamma,\delta} \). An advantage of this method is that the existing tools for performance evaluation and controller synthesis for conventional TEGs can be used for WBTEGs. Furthermore, we present a new algorithm for the calculation of the Kleene star for elements in \( \mathcal{E}^*[\delta] \), which is in general computationally more efficient than the algorithm presented in Cottenceau et al. (2014).

The paper is organized as follows: Section 2 summarizes the necessary facts on Petri nets, WBTEG and dioid theory. In Section 3, the modeling process of a WBTEGs in the dioid \( \mathcal{E}^*[\delta] \) is recalled. Section 4 introduces a decomposition method for elements in \( \mathcal{E}^*[\delta] \). Moreover, algorithms for the Kleene star
calculation in $\mathcal{E}^* [\delta]$ are compared. Finally, in Section 5, the controller design process for WBTEGs is illustrated.

2. WEIGHTED TIMED EVENT GRAPHS AND DI OIDS

2.1 Petri nets and Timed Event Graphs

In the following, we restate the necessary facts on Petri nets and TEGs (see, e.g., Baccelli et al. (1992) Cassandras and Lafontune (1999)). Matrices and vectors are indicated by bold letters. A Petri net graph is a directed bipartite graph $\mathcal{N} = (P, T, w)$, where:

- $P = \{p_1, p_2, \ldots, p_n\}$ is the finite set of places.
- $T = \{t_1, t_2, \ldots, t_m\}$ is the finite set of transitions.
- $w : (P \times T) \cup (T \times P) \to \mathbb{N}_0$ is the weight function.

$A := \{(p_i, t_j)|w(p_i, t_j) > 0\} \cup \{(t_j, p_i)|w(t_j, p_i) > 0\}$ is the arc set of the Petri net graph $\mathcal{N}$. A Petri net consists of a Petri net graph $\mathcal{N}$ and a vector of initial markings $M_0 \in (\mathbb{N}_0)^n$, i.e., an initial distribution of tokens over places in $\mathcal{N}$. The marking of the place $p_i$ is represented by $\phi_i \in \mathbb{N}_0$, and the marking of the transition $t_j$ by $\pi_j \in \{1, \ldots, n\}$.

Definition 1. A basic directed path from $t_i$ to $t_j$, i.e., $p_i \rightarrow t_j$, is defined as $p_i \rightarrow t_j \rightarrow p_k \rightarrow \ldots \rightarrow t_l \rightarrow p_l$. The initial marking $\pi_i = 1$ and the final marking $\pi_j = 1$.

Definition 2. For a Petri net $(\mathcal{N}, M_0, \phi)$, a basic directed path is an $\mathcal{N}$-arc set, i.e., an arc set in $\mathcal{N}$ that contains an $\mathcal{N}$-arc set with initial and final markings.

A WTEG is a directed graph with a marking function $\phi : P \to \mathbb{N}_0$, and a weight function $w : (P \times T) \cup (T \times P) \to \mathbb{N}_0$.

2.2 Weighted Timed Event Graphs

Definition 3. A vector of initial markings is a WTEG if $\phi_i = 1$ for all $i$. A WTEG is strongly connected if there exists a directed path from $t_i$ to $t_j$.

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2.3 Dioid Theory

A dioid $D$ is an algebraic structure with two binary operations, $\oplus$ (addition) and $\otimes$ (multiplication). Addition is commutative, associative, and idempotent (i.e., $\forall a \in D, a \oplus a = a$). The neutral element for addition, denoted by $e$, is absorbing for multiplication (i.e., $\forall a \in D, a \otimes e = e \otimes a = e$). Multiplication is associative, distributive over addition and has a neutral element denoted by $1$. Both operations can be extended to the matrix products.

The event-shift and time-shift operators $\Sigma$ and $\lambda$ can be interpreted as event-shift and time-shift operators, respectively.

Residuation theory is a formalism to address the problem of approximate mapping inversion over ordered sets, see Baccelli et al. (1992).

Theorem 1. (Baccelli et al. (1992)) A complete dioid $D$, $x = a^*b$ is the least solution of the implicit equation $x = axb$.

Residuation theory is a formalism to address the problem of approximate mapping inversion over ordered sets, see Baccelli et al. (1992).

Definition 5. (Residuation). Let $\mathcal{F}$ and $\mathcal{L}$ be ordered sets and $f : \mathcal{F} \to \mathcal{L}$ be an isomorphism, i.e., $a \leq b$ implies $f(a) \leq f(b)$. The mapping $f$ is said to be residuated if for all $y \in \mathcal{L}$, the least upper bound of the subset $\{x \in \mathcal{F}|f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of $f$.

Since a complete dioid is an ordered set, this notion is applicable for specific mappings defined over the dioid. For instance, on a complete dioid the mapping $R_a : x \mapsto xa$, (right multiplication) resp. $\Lambda_a : x \mapsto ax$ (left multiplication) are residuated. The residual mappings are denoted $R_a^\sharp(b) = b \preceq a = \{x \in \mathcal{F} | xa \preceq b\}$ (right division by $a$) resp. $\Lambda_a^\sharp(b) = a \preceq b = \{x \in \mathcal{F} | ax \preceq b\}$ (left division by $a$). In analogy to the extension of the product to the matrix case, we can extend left and right division to matrices with entries in a complete dioid.
2.4 Dioid $M^{\text{ord}}_{\text{in}}[\gamma, \delta]$

TEGs can be conveniently modeled via two-dimensional power series in $\gamma$ and $\delta$ with Boolean coefficients, where $\gamma$ and $\delta$ can be interpreted as event-shift and time-shift operators, respectively. Such a series is then given by $B[\gamma, \delta]$, and $M^{\text{ord}}_{\text{in}}[\gamma, \delta]$ is the quotient dioid of $B[\gamma, \delta]$ induced by the equivalence relation $x \equiv y$ is satisfied. Dynamic phenomena arising in TEGs can be described by the following additive basic operators:

For the modeling process of WBTEGs in $\Gamma$ with $\nu \in \mathbb{Z}$ and $\beta \in \mathbb{N}$, equipped with addition and multiplication defined by $\nu \in \mathbb{Z}$, and $\beta \in \mathbb{N}$ with $\nu, \beta \in \mathbb{N}$, and $\Gamma(\nu, \beta)$ are essential to describe the event-variant behavior of WBTEGS. Therefore, in the following we discuss them in detail.

Definition 6. (Dioid of E-operators $E$) We denote by $E$ the dioid of operators obtained by sums and compositions of operators in $\{\gamma^\nu, \beta^\nu, \mu^\nu, \beta\}$ with $\nu \in \mathbb{Z}$, and $\beta, m \in \mathbb{N}$, equipped with addition and multiplication defined by $\forall w, 1 \in E, \nu, \beta \in \mathbb{N}$, and $\Gamma(\nu, \beta)$ are essential to describe the event-variant behavior of WBTEGS. Therefore, in the following we discuss them in detail.

Proposition 1. (Cottenceau et al. (2014)). The basic E-operators $\{\gamma^\nu, \beta^\nu, \mu^\nu, \beta\}$ satisfy the following relations

$$\gamma^\nu \cdot \gamma^{\nu'} = \gamma^{
u + \nu'},$$

$$\mu^\nu \cdot \gamma^\nu = \gamma^{\nu + \mu},$$

$$\gamma^\nu \cdot \beta^\nu = \beta^{\nu + \nu}.$$  

Since E-operators only affect event numbering an E-operator $w$ can be described by a Counter-value to Counter-value (C/C) function $F_w : \mathbb{Z} \rightarrow \mathbb{Z}$. An input counter value $k_1 = x(\xi)$ is mapped to an output counter value $k_2$, for instance $F_{\mu^\nu, \beta^\nu}(k_1) = [k_1/b]m$. This follows immediately from the definition of $\mu^\nu, \beta^\nu$ in (5),(4). There is an isomorphism between the set of operators in $E$ and the set of (C/C) functions. The order relation over the dioid $E$ corresponds to the order induced by the min operation on (C/C) functions. For $F_w, F_v \in E$ with $w, v \in E$

$$w \leq v \iff \forall x \in \mathbb{Z}, (w(x)) \leq (v(x)) \iff \forall x \in \mathbb{Z}, (w(x)) \leq (v(x)) \iff F_w(x) \leq F_v(x) \iff k \in \mathbb{Z}.$$

3. WBTEGs AND THE DIOID $E^* [\delta]$

3.1 Dioids of Event Operators $E$

The dioid $E^* [\delta]$ introduced in Cottenceau et al. (2014) is suitable to model dynamic phenomena occurring in WBTEGs. For the modeling process of WBTEGs in $E^* [\delta]$, a counter function $x_1 : \mathbb{Z} \rightarrow \mathbb{Z}_\text{min}$, where $\mathbb{Z}_\text{min} = \mathbb{Z} \cup \{\infty, -\infty\}$, is associated to each transition $t_1$, $x_1(\xi)$ gives the accumulated number of events up to time $\xi$. A counter function is naturally a non-decreasing function, i.e., $x_1(\xi + 1) \geq x_1(\xi)$, and the set of counter functions is denoted by $\Sigma$. On $\Sigma$ addition $+$ and multiplication $\odot$ are defined as follows:

$$x, y \in \Sigma, (x + y)(\xi) := \min(x(\xi), y(\xi)),$$

$$\lambda \in \mathbb{Z}, (\lambda \odot x)(\xi) := \lambda + x(\xi).$$

The $+$ operation induces an order relation on $\Sigma$, i.e., for $x, y \in \Sigma, x \leq y \iff x + y = y$. An operator is a map $K : \Sigma \rightarrow \Sigma$ which is linear if (a) $\forall x, y \in \Sigma : K(x + y) = K(x) \odot K(y)$ and (b) $\lambda \odot K(x) = K(\lambda \odot x)$. An operator is additive if (a) is satisfied. Dynamic phenomena arising in TEGs can be described by the following additive basic operators:

$$\gamma \in \mathbb{Z}, \delta : \forall x \in \Sigma, (\delta x)(\xi) = x(\xi - \gamma),$$

$$\nu \in \mathbb{Z}, \gamma^\nu : \forall x \in \Sigma, (\gamma^\nu x)(\xi) = x(\xi + \nu),$$

$$b \in \mathbb{N}, \beta^b : \forall x \in \Sigma, (\beta^b x)(\xi) = [x(\xi)/b],$$

$$m \in \mathbb{N}, \mu^m : \forall x \in \Sigma, (\mu^m x)(\xi) = mx(\xi),$$

where $[a]$ is the greatest integer less than or equal to $a \in \mathbb{Q}$. The event-shift and time-shift operators $\gamma$ and $\delta$ are linear, whereas $\mu^m, \beta^b$ for $m \neq 1, b \neq 1$ are only additive. The three operators $\{\gamma^\nu, \beta^\nu, \mu^\nu, \beta\}$ are essential to describe the event-variant behavior of WBTEGS. Therefore, in the following we discuss them in detail.
\[
\mu_m \beta_b = \bigoplus_{i=0}^{n-1} \gamma^i \mu_{mn} \beta_{mb} \gamma^{(n-1-i)b}.
\] (9)

For instance with \( n = 3 \), the operator \( \mu_1 \beta_2 \) can be written as \( \mu_3 \beta_5 \gamma^4 \oplus \mu_4 \beta_6 \gamma^4 \oplus \mu_5 \beta_6 \gamma^4 \). Fig. 1 illustrates the extension of this \( \mu_1 \beta_2 \) operator. The intersection of the areas beneath \( F_{\mu_2 \beta_2} \gamma^4 \), \( F_{\gamma\mu_2 \beta_2} \gamma^4 \), and \( F_{\gamma^2 \mu_2 \beta_2} \) is equal to area beneath the (C/C) function \( F_{\mu_1 \beta_2} \).

![Fig. 1.](image-url)

**Proposition 3.** (Cotteneau et al. (2014)). Let \( w \in \mathcal{E} \), then
\[
\begin{align*}
\gamma^m \kappa w &= \gamma^n w, \\
\mu_m \kappa w &= \beta_m \gamma^{m-1} w, \\
\beta_b \kappa w &= \mu_b w, \\
\end{align*}
\] (10) (11) (12)

3.2 Dioid \( \mathcal{E}^*(\mathcal{E}[\delta]) \)

E-operators commute with the time-shift operator \( \delta^t \), i.e., \( \forall w \in \mathcal{E}, \delta^t w = w \delta^t \), Cotteneau et al. (2014). Therefore, one can define formal power series in \( \delta \) with exponents in \( \mathbb{Z} \) and coefficients \( w \in \mathcal{E} \) as follows.

**Definition 8.** (Dioid \( \mathcal{E}^*(\mathcal{E}[\delta]) \)) We denote by \( \mathcal{E}^*(\mathcal{E}[\delta]) \) the quotient dioid in the set of formal power series in one variable \( \delta \) with exponents in \( \mathbb{Z} \) and coefficients in the non commutative complete dioid \( \mathcal{E} \) induced by the equivalence relation \( \forall s \in \mathcal{E}^*(\mathcal{E}[\delta]), s = (\gamma^1)^s = s(\gamma^1)^s = s(\delta^{-1})^s \).

The subset of \( \mathcal{E}^*(\mathcal{E}[\delta]) \) obtained by restricting the coefficients to \( \mathcal{E}_{\text{per}} \), i.e., the set of periodic operators, is denoted by \( \mathcal{E}_{\text{per}}[\delta] \). A monomial in \( \mathcal{E}_{\text{per}}[\delta] \) is defined as \( w \delta^s \) where \( w \in \mathcal{E}_{\text{per}} \). In Cotteneau et al. (2014) it is shown that a monomial can be represented as
\[
\begin{align*}
\begin{pmatrix} \end{align*}
\] (13)

A polynomial in \( \mathcal{E}_{\text{per}}[\delta] \) is a sum of monomials \( p = \bigoplus_{i=1}^{K} w_i \delta^s \) such that \( \Gamma(w_i) = \Gamma(w_j) \) \( \forall i, j \in \{1, \ldots, K\} \). The gain \( \Gamma(p) \) of a polynomial is defined to be the gain of its coefficient, i.e., \( \Gamma(p) = \Gamma(w_i) \). A series \( s \in \mathcal{E}_{\text{per}}[\delta] \) is said to be ultimately periodic if it can be written as \( s = p \oplus q(\gamma^\delta)^s \), where \( n, \gamma \in \mathbb{N} \) and \( p, q \) are polynomials in \( \mathcal{E}_{\text{per}}[\delta] \) such that \( \Gamma(p) = \Gamma(q) \). The gain of \( s \) is then defined to be \( \Gamma(s) = \Gamma(p) \).

**Proposition 4.** (Cotteneau et al. (2014)). Let \( s_1, s_2 \) be two ultimately periodic series in \( \mathcal{E}_{\text{per}}[\delta] \) then:

- \( \Gamma(s_1) = \Gamma(s_2) \Rightarrow s_1 \oplus s_2 \) is an ultimately periodic series, with \( \Gamma(s_1 \oplus s_2) = \Gamma(s_1) \).
- \( s_1 \otimes s_2 \) (resp. \( s_1 \otimes s_1 \)) is an ultimately periodic series, with \( \Gamma(s_1 \otimes s_2) = \Gamma(s_1) \times \Gamma(s_2) \).
- \( \Gamma(s_1) = 1 \Rightarrow s_1^* \) is an ultimately periodic series, with \( \Gamma(s_1^*) = 1 \).

4. CORE REPRESENTATION OF A SERIES IN \( \mathcal{E}_{\text{per}}^*[\delta] \)

In this section we show how a series \( s \in \mathcal{E}_{\text{per}}^*[\delta] \) can be decomposed into a \( \mathcal{M}_{\text{per}}^{\mathcal{E}}[\gamma, \delta] \) matrix (core) combined with an input column vector and an output row vector. Thanks to this representation all operations between series \( s \in \mathcal{E}_{\text{per}}^*[\delta] \) can be equally described by operations between their corresponding \( \mathcal{M}_{\text{per}}^{\mathcal{E}}[\gamma, \delta] \) core matrices.

4.1 Matrix Notations

In the following we define some matrices frequently used throughout this paper. In the core representation the input interface is a column vector defined as
\[
\beta_b := [\beta_b \gamma^{b-1} \cdots \beta_b \gamma \beta_b]^T.
\]

In analogy we define an output interface as a row vector
\[
\mu_m := [\mu_m \gamma \mu_m \cdots \mu_m \gamma^{m-1} \mu_m].
\]

The index \( b \) (resp. \( m \)) determines the division (resp. multiplication) coefficient and gives the dimension of the vector. The identity matrix, denoted by \( I \), and the zero matrix, denoted by \( \varepsilon \), are given by
\[
I := \begin{bmatrix} e & \varepsilon & \cdots & \varepsilon \\
\varepsilon & \cdots & \varepsilon & \varepsilon \\
\vdots & \ddots & \ddots & \ddots \\
\varepsilon & \cdots & \varepsilon & \varepsilon \end{bmatrix}, \quad \varepsilon := \begin{bmatrix} \varepsilon & \cdots & \varepsilon \end{bmatrix}.
\]

Finally, we define a particular square matrix,
\[
E := \begin{bmatrix} e & \gamma^1 & \cdots & \gamma^1 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
e & \cdots & \cdots & e \end{bmatrix},
\]

which is composed of operators \( \{e, \gamma^1\} \). If necessary, the dimension of \( E \) is stated as an index, e.g., \( E_{mn} \in \{e, \gamma^1\}^{m \times m} \).

4.2 Core-Form of a Series in \( \mathcal{E}_{\text{per}}^*[\delta] \)

The decomposition of a series in \( \mathcal{E}_{\text{per}}^*[\delta] \) is carried out according to the following equation
\[
s = \mu_m X \beta_b.
\] (14)

This equation is called core-equation. Recall that \( \mathcal{M}_{\text{per}}^{\mathcal{E}}[\gamma, \delta] \) is a subdioid of \( \mathcal{E}^*[\delta] \). We say \( Q \in \mathcal{M}_{\text{per}}^{\mathcal{E}}[\gamma, \delta]^{m \times b} \) is a core of \( s \in \mathcal{E}_{\text{per}}^*[\delta] \), if \( Q \) is a solution of (14), i.e., \( s = \mu_m Q \beta_b \). Note that the dimension of the core matrix \( Q \in \mathcal{M}_{\text{per}}^{\mathcal{E}}[\gamma, \delta]^{m \times b} \) is determined by the periodicity \( (m, b) \) of the series \( s \). In general, there exist several cores \( Q \) which solve (14). In the following, we show how to obtain a core \( Q \) for a given series \( s \in \mathcal{E}_{\text{per}}^*[\delta] \). Furthermore, we prove that \( s \) admits a unique greatest core, denoted \( Q \in \mathcal{M}_{\text{per}}^{\mathcal{E}}[\gamma, \delta]^{m \times b} \) (greatest with respect to the order relation in the dioid \( \mathcal{M}_{\text{per}}^{\mathcal{E}}[\gamma, \delta] \), i.e., \( Q \geq Q' \)).

Due to (13) and by using (7) \( \gamma^{n/m} \mu_m = \gamma^{n-[n/m]} \mu_m \gamma^{[n/m]} \) (resp. \( \beta_b \gamma^{n/m} = \gamma^{(n/b)} \beta_b \gamma^{(n/b)-[n/b]} \)), a series \( s = p \oplus q(\gamma^\delta)^s \in \mathcal{E}_{\text{per}}^*[\delta] \) can be expressed as
Consider an input interface $\beta_i$ and an output interface $\mu_i$ of the same size. Then, the scalar product $\mu_i\beta_i$ is the identity $e$, since \((9)\),

\[
\mu_i \otimes \beta_i = \mu_i \beta_i = \gamma_{-1}^i + \gamma_{-2}^i \beta_i + \cdots + \gamma_{-N}^i \beta_i = e. 
\]

The dyadic product $\beta_i \otimes \mu_i$ is the $E$ matrix of size $i \times i$,

\[
\beta_i \otimes \mu_i = \begin{bmatrix} 
\beta_i \gamma_{-1}^i \mu_i & \beta_i \gamma_{-1}^i \mu_i & \cdots & \beta_i \gamma_{-2}^i \mu_i \\
\beta_i \gamma_{-2}^i \mu_i & \beta_i \gamma_{-1}^i \mu_i & \cdots & \beta_i \gamma_{-3}^i \mu_i \\
\vdots & \vdots & \ddots & \vdots \\
\beta_i \gamma_{-i}^i \mu_i & \beta_i \gamma_{-i+1}^i \mu_i & \cdots & \beta_i \gamma_{-N}^i \mu_i \end{bmatrix} = E. 
\]

Proposition 5. For the $E$ matrix the following relations hold $E \otimes E = E$; $E \otimes \beta_i = \beta_i$; $\mu_i \otimes E = \mu_i$.

Proof.

\[
E \otimes E = \beta_i \otimes \mu_i \otimes \beta_i \otimes \mu_i = \beta_i \otimes e \otimes \mu_i = E, 
\]

\[
E \otimes \beta_i = \beta_i \otimes \mu_i \otimes \beta_i = \beta_i \otimes e = \beta_i, 
\]

\[
\mu_i \otimes E = \mu_i \otimes \beta_i \otimes \mu_i = e \otimes \mu_i = \mu_i. 
\]

Proposition 6. For $D \in \mathcal{E}^*[\delta]^{1 \times n}$ and $P \in \mathcal{E}^*[\delta]^{n \times 1}$, we have

\[
\mu_i \cdot D = \beta_i \otimes D, 
\]

\[
P \beta_i = P \otimes \mu_i. 
\]

For $M \in \mathcal{E}^*[\delta]^{n \times m}$, $N \in \mathcal{E}^*[\delta]^{m \times n}$, $M = ME$ and $\widetilde{N} = EN$, we have

\[
\beta_i \otimes \mu_i = \widetilde{M} \otimes \beta_i, 
\]

\[
\beta_i \otimes \widetilde{N} = \mu_i \otimes \widetilde{N}. 
\]

Proof. See Appendix B.

Proposition 7. Let $s = \mu_m Q \beta_b \in \mathcal{E}_{\text{per}}[\delta]$, the core equation $s = \mu_m X \beta_b$ has a unique greatest solution, denoted $\hat{Q}$ and given by

\[
\hat{Q} = \mu_m Q E_b. 
\]

Proof. Consider the inequality $\mu_m \hat{X} \beta_b \leq \mu_m Q \beta_b = s$. The greatest solution for $\hat{X}$ is

\[
\mu_m \hat{X} \beta_b = \mu_m Q \beta_b \beta_b = \mu_m Q \beta_b = s. 
\]

Furthermore, we can check that $\hat{Q}$ solves the core equation. Since $\mu_m = \mu_m E_m$ and $\beta_b = E_b \beta_b$, $\mu_m \hat{Q} \beta_b = \mu_m E_m Q E_b \beta_b = \mu_m Q \beta_b = s$. 

\[
\mu_m \hat{Q} \beta_b = \mu_m E_m Q E_b \beta_b = \mu_m Q \beta_b = s. 
\]
Remark 1. The greatest core matrix \( \hat{Q} \) has the following properties. Since: \( E \otimes E = E, E\hat{Q} = E\hat{Q}E = \hat{Q} \), \( \hat{Q}E = EQEE = \hat{Q} \).

Example 4. The greatest core of the series considered in Example 3 is given by
\[
\hat{Q} = E_3 Q E_2,
\]
Similarly to the extension of the basic \( \mu_m \beta_b \) operator in (9) we can extend matrix \( \hat{Q} \) in the core-form of a series.

Proposition 8. A series \( s = \mu_m \beta_b \in \mathcal{E}_\per^\star[\beta] \) can be expressed with a multiple period \( (nm, nb) \) by extending the core matrix \( \hat{Q} \), i.e., \( s = \mu_m \beta_b \in \mathcal{M}_m^\star[\gamma] \), \( \hat{Q}' \in \mathcal{M}_m^\star[\gamma] \), \( \hat{Q}' \) and is given by
\[
\hat{Q}' = \begin{bmatrix}
\beta n\gamma n - 1 \hat{Q} \mu n \cdots \beta n\gamma n - 1 \hat{Q} \gamma n - 1 \mu n \\
\beta n \hat{Q} \mu n \cdots \beta n \hat{Q} \gamma n - 1 \mu n
\end{bmatrix}.
\]

Proof. See Appendix B.

4.3 Operations between Core Matrices

Because of Prop. 8, by extending the core-form if necessary, two series with equal gain can be expressed with their least common period.

Proposition 9. (Sum of series).

Let \( s = \mu_m \beta_b \), \( s' = \mu_m \beta_b' \in \mathcal{E}_\per^\star[\beta] \), the sum \( s \oplus s' = \mu_m \beta_b \), where \( \hat{Q}'' = (\hat{Q} \oplus \hat{Q}') \) is again a greatest core.

Proof.
\[
\begin{align*}
\mu_m \hat{Q} \beta_b & \oplus \mu_m \hat{Q} \beta_b' = \mu_m (E \hat{Q} \oplus E \hat{Q}') \beta_b \\
& = \mu_m \left( E \hat{Q} E \hat{Q} E \hat{Q} \right) E \beta_b \\
& = \mu_m \left( E \hat{Q} E \hat{Q} \right) E \beta_b
\end{align*}
\]

Proposition 10. (Product of series).

Let \( s = \mu_m \beta_b \in \mathcal{E}_\per^\star[\beta] \) and \( s' = \mu_b \beta_b' \in \mathcal{E}_\per^\star[\beta] \), the product \( s \otimes s' = \mu_m \beta_b \), where \( \hat{Q}'' = \hat{Q} \hat{Q}' \) is again a greatest core.

Proof.
\[
\begin{align*}
\mu_m \hat{Q} \beta_b \mu_b \hat{Q} \beta_b' & = \mu_m \hat{Q} \beta_b \mu_b \hat{Q} \beta_b' \\
\text{Furthermore: } \hat{Q} \hat{Q}' & = E \hat{Q} E \hat{Q} E = \hat{Q}''
\end{align*}
\]

Proposition 11. (Kleene star of series). Let \( s = \mu_m \hat{Q} \beta_m \in \mathcal{E}_\per^\star[\beta] \). Then, \( s^\star \) can be obtained by
\[
s^\star = \mu_m \hat{Q} \beta_m.
\]

Proof. In this case, \( I(s) = 1 \) means that \( \hat{Q} \) is a square matrix.
\[
s^\star = e \oplus \mu_m \hat{Q} \beta_m \oplus \mu_m \hat{Q} \beta_m \mu_m \hat{Q} \beta_m \oplus \cdots
\]
Since: \( e = \mu_m \beta_m \) (15), \( E = \beta_m \mu_m \) (16) and \( E \hat{Q} = \hat{Q} \) (Remark 1),
\[
\begin{align*}
& = \mu_m \beta_m \oplus \mu_m \hat{Q} \beta_m \oplus \mu_m \hat{Q} \beta_m \beta_m + \cdots \\
& = \mu_m (I \oplus \hat{Q} \oplus \hat{Q} \oplus \cdots) \beta_m \\
& = \mu_m \hat{Q}^\star \beta_m
\end{align*}
\]

4.4 Comparison of Kleene Star Algorithms

In Cottenceau et al. (2014) it is shown that the Kleene star of a polynomial, \( p = \bigoplus_{i=1}^l \gamma^n \mu_m \beta_m \gamma^n \delta^n \in \mathcal{E}_\per^\star[\gamma] \), can be obtained by the following computation
\[
p^\star = e \oplus \left( \bigoplus_{i=1}^l \bigoplus_{j=1}^l (\psi^i) \gamma^n \mu_m \beta_m \gamma^n \delta^n \right),
\]
where \( \psi \in \mathcal{M}_n^\star[\gamma, \delta] \) is a matrix such that \( \forall a, b \in \{1, \ldots, I\}, \psi_{ab} = \gamma^n \nu \). For large \( I \) this algorithm is computationally very expensive since it scales with the number \( I \) of monomials in the polynomial. First, we have to calculate the star of \( \psi \in \mathcal{M}_n^\star[\gamma, \delta] \). Second, we have the sum of \( I^2 \) ultimately periodic series \( s \in \mathcal{E}_\per^\star[\gamma] \). The approach presented in Prop. 11 is in general computationally less expensive, since we only have to calculate the star of the core matrix \( \hat{Q} \) which is of size \( m \times m \) (determined by the period). Roughly speaking, if the number of monomials in the polynomial is bigger than the period of the polynomial, then the algorithm suggested in Prop. 11 is preferable.

4.5 Division in \( \mathcal{E}_\per^\star[\gamma] \)

Proposition 12. Let \( s = \mu_m \beta_b \), \( s' = \mu_m \beta_b' \) be ultimately periodic series in \( \mathcal{E}_\per^\star[\gamma] \). Then,
\[
s \triangleleft s' = \mu_b (\hat{Q} \triangleleft \hat{Q} \beta_b),
\]

Proof. since (A.1) and \( \mu_m \gamma (\mu_m \hat{Q}) = \beta_m \mu_m \hat{Q} = \hat{Q} \),
\[
(\mu_m \beta_b') \triangleleft (\mu_m \beta_b) = (\hat{Q} \beta_b') \triangleleft (\mu_m \beta_b) = (\hat{Q} \beta_b'),
\]

Since \( \hat{Q}_b \beta_b = \hat{Q}_m \beta_b \) see Prop. 6. (A.1) and (A.2),
\[
\begin{align*}
(\hat{Q} \beta_b') & \triangleleft (\hat{Q} \beta_b) = \beta_b \triangleleft (\hat{Q} \beta_b) \\
& = (\hat{Q} \beta_b') \triangleleft (\hat{Q} \beta_b) = \beta_b \triangleleft (\hat{Q} \beta_b)
\end{align*}
\]
Proposition 13. Let \( s = \mu_m \beta_b \), \( s' = \mu_m \beta_b' \) be ultimately periodic series in \( \mathcal{E}_\per^\star[\gamma] \). Then,
\[
s \triangleleft s' = \mu_m (\hat{Q} \triangleleft \hat{Q}' \beta_b).\]

Proof. The proof is analogous to the proof of Prop. 12.

Additionally, it can be shown that \( \hat{Q} \triangleleft \hat{Q} \) and \( \hat{Q} \triangleleft \hat{Q} \) are greatest cores.

5. MODELING AND CONTROL OF SISO WBTEGs

5.1 Transfer Matrices of SISO WBTEGs

From now on we assume that WBTEGs are operating under the earliest functioning rule. Then the firing relation between two
transitions, composing a basic directed path $t_j \rightarrow p_i \rightarrow t_o$, can be described by $E$-operators combined with the time-shift operator $\delta^t$ as follows: $x_o = \beta w(p_i,t_o) \delta^t(\phi) \gamma(M_0), \mu w(t_j,p_i)x_j$, where $x_j$ (resp. $x_o$) refers to the counter function of transition $t_j$ (resp. $t_o$), $w(p_i,t_o)$ (resp. $w(t_j,p_i)$) is the weight of the arc $(p_i,t_o)$ (resp. $(t_j,p_i)$), $(\phi)$ is the holding time of place $p_i$ and $(M_0)$ is the initial marking of $p_i$. For instance, take the path $t_2 \rightarrow p_3 \rightarrow t_3$ of the WBTEG in Fig. 2, the firing relation between $t_2$ and $t_3$ corresponds to an operator representation $x_3 = \beta_2 \delta^t \gamma^1 \mu_1 x_2$. Therefore, the firing relation between internal, input and output transition in a WBTEG can be described by a state space representation:

$$x = Ax \otimes Bu, \quad y = Cx,$$

where $x$ (resp. $u,y$) refers to counter functions of internal (resp. input, output) transitions and $A \in E_{per}[\delta]^{n \times n}; B \in E_{per}[\delta]^{n \times m}, C \in E_{per}[\delta]^{p \times n}$.  

**Theorem 3.** (Cottenceau et al. (2014)). For an $m$-inputs and $p$-outputs WBTEG, the entries of the transfer matrix $H = CA^*B$ are ultimately periodic series in $E_{per}[\delta]$. 

**Example 5.** Consider the WBTEG in Fig. 2. The transfer function $h_s$ describes the firing relation between input transition $t_1$ and output transition $t_3$ and is given by

$$h_s = \beta_2 \gamma^1 (\gamma^3 \delta^1)^* \mu_3 = (\mu_3 \beta_2 \gamma^1 + \gamma^2 \mu_3 \beta_2) \delta^1 (\gamma^1 \delta^1)^*,$$

$$= \mu_3 Q \beta_2 = \mu_3 \left[ \begin{array}{c} \delta^1 (\gamma^1 \delta^1)^* \\ \gamma^1 \delta^1 (\gamma^1 \delta^1)^* \\ \gamma^1 \delta^2 (\gamma^1 \delta^1)^* \\ \gamma^1 \delta^2 (\gamma^1 \delta^1)^* \end{array} \right] \beta_2.$$ 

Since, $\hat{Q} \in M_{in}^{\alpha} [\gamma, \delta]^{m \times m}$ is a matrix in $M_{in}^{\alpha} [\gamma, \delta]$ we can use the existing tools introduced in Maia et al. (2003) for the feedback design process of WBTEGs.

**Example 6.** Consider the WBTEG in Fig. 2, the greatest neutral output feedback is

$$f = \mu_2 (Q \hat{Q} \hat{Q} \beta_3),$$

$$= \mu_2 \left[ \begin{array}{c} \gamma^1 \delta^1 (\gamma^1 \delta^1)^* \\ \gamma^1 \delta^2 (\gamma^1 \delta^1)^* \\ \gamma^1 \delta^2 (\gamma^1 \delta^1)^* \end{array} \right] \beta_3.$$ 

We can express this feedback $f$ as an equivalent series in $E_{per}[\delta]$.  

$$f = (\gamma^1 \delta^1)^* (\gamma^2 \mu_2 \beta_3 + \gamma^1 \mu_2 \beta_3 \gamma^1).$$

Fig. 3 illustrates the closed loop system. The feedback bounds the amount of tokens in place $p_1, p_3$ while the throughput of the system is preserved. The feedback of this example is computed with help of the software tools Hardouin et al. (2009).

![Fig. 3. Closed loop system.](image)

### 6. CONCLUSION

In this work we present the algebraic tools to address the problem of "just-in-time" feedback synthesis for the class of SISO WBTEGs. The introduced method is based on a decomposition of the transfer function of a SISO WBTEG such that the dynamic of the system can be expressed in an event-invariant core matrix in $M_{in}^{\alpha} [\gamma, \delta]$. An advantage of this method is that the already existing tools for feedback synthesis for conventional TEGs can be directly applied to SISO WBTEGs (a library for manipulating series in $M_{in}^{\alpha} [\gamma, \delta]$ is available in Hardouin et al. (2009)). The generalization of the problem to multiple-input and multiple-output (MIMO) WBTEGs will be investigated in further work.

### Appendix A. FORMULAS OF RESIDUATION

$$(ab) \cdot x = b(yx) \quad x(ya) = (yx)a \quad (a \cdot yb) = (ay)b$$

(A.1)

$$(a \cdot bx) = y(xb) \quad a \cdot (xby) = (a \cdot yb)$$

(A.2)

### Appendix B. PROOFS

**Proof.** of Prop. 6. (17) for the left product, by definition of the residual mapping $\mu_m: D$ is the greatest solution of the following inequality

$$f = \mu_2 (Q \hat{Q} \hat{Q} \beta_3),$$

$$= \mu_2 \left[ \begin{array}{c} \gamma^1 \delta^1 (\gamma^1 \delta^1)^* \\ \gamma^1 \delta^2 (\gamma^1 \delta^1)^* \\ \gamma^1 \delta^2 (\gamma^1 \delta^1)^* \end{array} \right] \beta_3.$$ 

We can express this feedback $f$ as an equivalent series in $E_{per}[\delta]$.  

$$f = (\gamma^1 \delta^1)^* (\gamma^2 \mu_2 \beta_3 + \gamma^1 \mu_2 \beta_3 \gamma^1).$$

Fig. 3 illustrates the closed loop system. The feedback bounds the amount of tokens in place $p_1, p_3$ while the throughput of the system is preserved. The feedback of this example is computed with help of the software tools Hardouin et al. (2009).  

![Fig. 3. Closed loop system.](image)
This matrix inequality can be transformed into a set of $n$ inequalities,
$$
\mu_m x_{11} + \gamma^1 \mu_m x_{21} + \cdots + \gamma^{m-1} \mu_m x_{1n} \leq d_1,
$$
$$
\vdots
$$
$$
\mu_m x_{1n} + \gamma^1 \mu_m x_{2n} + \cdots + \gamma^{m-1} \mu_m x_{mn} \leq d_n.
$$
Because of Prop. 3, for each inequality $i \in \{1, \cdots, n\}$ we obtain
$$
x_{1i} \leq \mu_m d_i = \beta_{i} \gamma^{m-1} d_i,
$$
$$
\vdots
$$
$$
x_{ni} \leq (\gamma^{m-1} \mu_m) d_i = \beta_{i} d_i.
$$
Rewriting the inequalities into matrix form leads to
$$
X \leq \mu_m \gamma D = \begin{bmatrix}
\beta_{m} \\
\vdots \\
\beta_{m}
\end{bmatrix}
D = \beta_m \otimes D.
$$
For the right product by $\beta_b$ we have
$$
X \beta_b \preceq P \Leftrightarrow X \preceq P \beta_b,
$$
where $X$ is of size $n \times m$ and $P$ is of size $n \times 1$. We obtain for each $i \in \{1, \cdots, n\}$ the following inequalities
$$
x_{1i} \leq p_i (\beta_b \gamma^{b-1}) = p_i \mu_b,
$$
$$
\vdots
$$
x_{ni} \leq p_i \beta_b \mu_b = p_i \gamma^{b-1} \mu_b.
$$
This can be expressed in matrix form
$$
X \leq P \beta_b = P [\mu_1, \gamma^1 \mu_b, \cdots, \gamma^{b-1} \mu_b] = P \otimes \mu_b.
$$
To prove (18), since $\beta \mu E = E = EE$ and due to (17) $P \mu = P \beta$, we can write
$$
\tilde{M} \beta \mu_i = (\tilde{M} \beta_i) \mu_i = (\tilde{M} \beta_i) \beta_i \mu_i.
$$
Since, $(x \beta \gamma^b y) \beta = x \beta (y \gamma^b)$ (A.1) and $\mu_i \beta = \gamma^{i-1}$ (see 15),
$$
(\tilde{M} \beta_i) \beta_i \mu_i = (\tilde{M} \beta_i) \gamma (\mu_i \beta_i) = (\tilde{M} \beta_i) \gamma \epsilon = \tilde{M} \beta_i.
$$
The proof of $\beta \gamma N = \mu \otimes \tilde{N}$ is analogous. $\square$

**Proof.** of Prop. 8. We can extend the core matrix $Q$ of a series, i.e.,
$$
s = \mu_n Q \beta_b = \mu_n \mu_m Q \beta_b \mu_n \beta_{nb}.
$$
Since, $\beta_n \gamma^{m-1} = \beta_n \gamma^{m-1} \beta_m \gamma^{m-1} = \beta_n \gamma^{m-1} \beta_m \gamma^{m-1}$ then
$$
\beta_{nm} = \begin{bmatrix}
\beta_n \gamma^{m-1} \\
\vdots \\
\beta_n \gamma^{m-1}
\end{bmatrix}
\begin{bmatrix}
\beta_n \\
\vdots \\
\beta_n
\end{bmatrix} = \begin{bmatrix}
\beta_n \gamma^{m-1} \\
\vdots \\
\beta_n \gamma^{m-1}
\end{bmatrix}.
$$
This leads to
$$
\beta_{nm} \mu_n = \begin{bmatrix}
\beta_n \gamma^{m-1} E \\
\vdots \\
\beta_n E
\end{bmatrix}.
$$
Respectively $\beta_b \mu_{nb}$ is given by
$$
\beta_b \mu_{nb} = [E \mu_n \cdots E \gamma^{n-1} \mu_n].
$$
Finally we obtain
$$
Q' = \begin{bmatrix}
\beta_n \gamma^{n-1} E \\
\vdots \\
\beta_n E
\end{bmatrix} Q [E \mu_n \cdots E \gamma^{n-1} \mu_n],
$$
$$
= \begin{bmatrix}
\beta_n \gamma^{n-1} \hat{Q} \mu_n \\
\vdots \\
\beta_n \gamma^{n-1} \hat{Q} \gamma^{n-1} \mu_n
\end{bmatrix}.
$$
The extended core is a matrix in $M_{nm}^{\gamma \nu}$, since $\beta_n \gamma^{\nu} \mu_n = \gamma^{\nu / n}$. Furthermore, the extended core $Q'$ is a greatest core since
$$
\hat{Q}' = \begin{bmatrix}
E \beta_{nm} \mu_n Q \beta_b \mu_{nb} E \\
\vdots \\
E \beta_{nm} \mu_n Q \beta_b \mu_{nb} \mu_{nb}
\end{bmatrix} = \begin{bmatrix}
\beta_{nm} \mu_n Q \beta_b \mu_{nb}
\end{bmatrix} = Q'.
$$

REFERENCES


