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CONSENSUS FOR AGENTS WITH DOUBLE INTEGRATOR DYNAMICS IN HETEROGENEOUS NETWORKS

D. Goldin and J. Raisch

ABSTRACT

This paper studies the convergence properties of consensus algorithms for agents with double integrator dynamics communicating over networks modelled by undirected graphs. The positions and velocities of the agents are shared along heterogeneous, *i.e.* different, undirected communication networks. The main result is that consensus can be achieved, even though the networks along which position and velocity information are shared are different, and not even connected. Insights on the consensus rate are given based only on the topological properties of the network.

Key Words: Multi-agent systems, second-order consensus, heterogeneous networks, undirected networks, double integrator dynamics.

I. INTRODUCTION

Consensus algorithm is the umbrella term for any algorithm that results in a number of autonomous mobile agents agreeing on a state variable using only local information. The widest-known single integrator consensus algorithm was first introduced in [1] in the context of sociological networks. Later consensus reemerged in the context of distributed computing. This work led to [2] and has since been an active research area within the control community. A lot of consensus applications can be found in [3,4] and the references therein.

Most existing literature addresses the case of agents governed by single integrator dynamics. However, many real life applications possess higher order dynamics. Particularly in the area of autonomus vehicles it is often desirable to achieve consensus using not only information on the agents' positions but also on their velocities. For example, some mobile robots can be feedback linarised and then described as having double integrator dynamics, which naturally leads to an extended algorithm that uses the additional state information.

An algorithm for double integrator dynamics has been proposed by [5]. Recently, a lot of work has focused on double integrator consensus. This can be roughly separated in two groups: [6–11] assume that both velocity and position information can be measured and communicated in the same way, resulting in homogeneous communication networks. On the other hand, [12,13] assume that there is no velocity information at all. In this case, the system in [5] becomes unstable and consensus is reached by data sampling or introducing delays in information exchange. Reference [14] studies double integrator consensus modelled by a Markov process.

In our work [15], different assumptions are made. Particularly, we believe that communication networks are rarely homogeneous and we study the case where velocity and position information is shared along different, and possibly disconnected, networks, making the overall network heterogeneous. This is motivated by application: agents that measure and communicate only one of their states are cheaper both in terms of hardware and communication costs. Furthermore, even if the networks were assumed to be homogeneous, information loss may create a heterogeneity. Homogeneous networks can be treated as a special case of heterogeneous networks, and in fact the results in [5] still hold in this context.

In the current paper, we study the algorithm introduced in [5] under different communication networks for velocity and position information. We thus generalize the existing work to heterogeneous communication topologies and show conditions under which it achieves consensus. Our main result is that consensus on velocities can be achieved even if the networks are disconnected. The results are given for undirected graphs. The present work is an extended and corrected version of [15].

The article is organised as follows. In Section II we give the theoretical background of our work, restating some basic results in graph theory and matrix polynomial theory. In Section III we introduce our model and the algorithm used. Section IV contains the main result on the convergence of the consensus algorithm in heterogeneous networks. In Section V

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D. Goldin is with the Control Systems Group of TU Berlin, Einsteinufer 11, 10587 Berlin, Germany.

J. Raisch is with the Control Systems Group of TU Berlin and with Max-Planck-Institut für Dynamik komplexer technischer Systeme, Sandtorstr. 1, 39106 Magdeburg, Germany.

D. Goldin is the corresponding author (e-mail: goldin@control.tu-berlin.de).

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we study the convergence rate of the algorithm and its dependence on the graph structure. Finally, in Section VI we give an outlook on our current work on consensus for double integrator systems over directed graphs.

II. PRELIMINARIES

Throughout this paper we write $I^{m\times m}$ for the $m \times m$ identity matrix and $1^{k\times m}$ and $0^{k\times m}$ for the one and zero matrix of size $k \times m$, respectively. We write lowercase latin letters (*e.g.* x) for vectors. $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ denote the real and imaginary parts of a complex number, respectively. The conjugate transpose of a vector v is denoted by v^*. We use greek letters (*e.g.* λ) to denote eigenvalues and order the eigenvalues according to $|\mathrm{Re}(\lambda_1)| \le |\mathrm{Re}(\lambda_2)| \le \ldots \le |\mathrm{Re}(\lambda_m)|$. The spectrum of a matrix L is denoted by spec(L). We write the Jordan canonical form of L as $\mathcal{J}(L)$. The number of Jordan blocks of L corresponding to the eigenvalue λ is $j_L(\lambda)$, while $|j_{L,i}(\lambda)|$, $1 \le i \le j_L(\lambda)$ denotes the size of the ith Jordan block of L corresponding to λ. We order the Jordan blocks of an eigenvalue according to their size, $|j_{L,1}(\lambda)| \ge |j_{L,2}(\lambda)| \ge \ldots \ge |j_{L,j_L(\lambda)}(\lambda)|$. We reserve n for the number of agents in the formation.

The high-level properties of the communication topology can be modelled by a communication graph. In order to make this paper self-contained we now present some existing definitions and results in algebraic graph theory and matrix polynomial theory. In this section all results provided without proof are taken from the respective literature.

2.1 Graph theory

A standard book on graph theory is, *e.g.*, [16]. Let us briefly recap the notions that will be used in this article.

A graph is generally given by a tuple $G = (V, E)$. Herein, the set of nodes is given by $V = \{v_1, v_2, \ldots, v_n\}$, and a node represents an individual agent. The set of edges is $E \subseteq V \times V$. An edge $\varepsilon_{ij} \in E$ between two nodes signifies that v_i may send information to v_j. $N(v_i)$ denotes the neighborhood of v_i, *i.e.* all v_j such that $\varepsilon_{ji} \in E$.

We assume that the graphs are undirected, *i.e.* $\varepsilon_{ij} \in E \Leftrightarrow \varepsilon_{ji} \in E$ and that they contain no self-loops, *i.e.*, that there is no edge ε_{ii}.

A k-partition of V is given by $\pi = (V_1, \ldots, V_k)$, $V_i \subseteq V$, such that each node belongs to exactly one V_i. The characteristic matrix of a k-partition is given by $Q \in \mathbb{R}^{n\times k}$, where $q_{ij} = 1$ if $v_i \in V_j$ and $q_{ij} = 0$ otherwise. The partition is called almost equitable [17] if $\forall i, j \in \{1, \ldots, k\}$ with $i \ne j$, $\forall v \in V_i$: $|N(v) \cap V_j| = d_{ij}$, $d_{ij} \in \mathbb{N}_0$.

The union of graphs $G_i = (V, E_i)$ is defined as $G := (V, \cup_i E_i)$. A path in a graph is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence are connected by an edge.

An undirected graph is connected if there exists a path between any two nodes. It is disconnected otherwise. If the graph is disconnected, then it has several connected components. The extreme case is the graph with no edges, $E = \emptyset$, which has n connected components. The converse case, $E = \{V \times V \setminus \cup_i \varepsilon_{ii}\}$, is called a fully connected graph.

An important property of graph structures is that they have a matrix representation, among them the adjacency matrix $A(G)$, with entries $\alpha_{ji} = 1$ if an edge from v_i to v_j exists and $\alpha_{ji} = 0$ otherwise.

The degree of a node is given by $d(v_i) = \Sigma_j \alpha_{ij}$. The graph Laplacian $L(G)$ is given as $L(G) = \mathrm{diag}(d(v_i)) - A(G)$. When clear, we will write L instead of $L(G)$. The Laplacian matrix of an undirected graph has the following properties:

- L is symmetric positive semi-definite and therefore has n linearly independent eigenvectors, all nonzero eigenvalues of L are positive and real, and the left and right eigenvectors coincide,
- the number of zero eigenvalues is the number of connected components of the graph, *i.e.* L has exactly one zero eigenvalue if the graph is connected (Matrix Tree Theorem),
- if G has k connected components, then its nodes can be renamed such that L has block diagonal form with k blocks, where each of the blocks is a Laplacian matrix
- the rows and columns of L sum up to 0, *i.e.* $1^{n\times 1}$ is a right eigenvector of L corresponding to a zero eigenvalue,

Lemma 1. Let L be an $n \times n$ Laplacian matrix associated with a connected undirected graph, let $b = (b^1, \ldots, b^n)^T$ be a real vector with entries $b^i \ge 0$, $b \not\equiv 0$. There is no vector v that satisfies $Lv = \pm b$.

Proof. Suppose that such a vector existed. Multiply both sides of $Lv = \pm b$ by $1^{1\times n}$ from the left. Knowing that the column sums of L are zero we obtain $0 = \pm\sum_{i=1}^{n} b^i$, which is a contradiction. □

Particularly there is no v satisfying $Lv = 1^{n\times 1}$.

Lemma 2. Let L be an $n \times n$ Laplacian matrix associated with a connected undirected graph. The vector $v = (1^{1\times k}, 0^{1\times m})^T$, $k + m = n$, $k, m \ne 0$, is not an eigenvector of L.

Proof. Suppose that v is an eigenvector of L to the eigenvalue λ. Write $L = \begin{pmatrix} L_1^{k\times k} & L_2 \\ L_2^T & L_3^{m\times m} \end{pmatrix}$. Then $Lv = \lambda v$, *i.e.*, $\begin{pmatrix} L_1 1^{k\times 1} \\ L_2^T 1^{k\times 1} \end{pmatrix} = \lambda \begin{pmatrix} 1^{k\times 1} \\ 0^{m\times 1} \end{pmatrix}$. This implies $L_2^T 1^{m\times 1} \equiv 0$. This is satisfied if and only if $L_2^T \equiv 0$, as all offdiagonal elements of a Laplacian are nonpositive. This is a contradiction to the requirement that L is connected. □

Suppose that G consists of k connected components of size $k_1, k_2, \ldots, k_k$. L, therefore, can be presented in block diagonal form. A basis of the kernel of L can then be given by the k vectors

$$\left\{\begin{pmatrix} 1^{k_1\times 1} \\ 0^{(n-k_1)\times 1}\end{pmatrix}, \begin{pmatrix} 0^{k_1\times 1} \\ 1^{k_2\times 1} \\ 0^{n_1\times 1}\end{pmatrix}, \ldots, \begin{pmatrix} 0^{n_2\times 1} \\ 1^{k_{k-1}\times 1} \\ 0^{k_k\times 1}\end{pmatrix}, 1^{n\times 1}\right\}, \tag{1}$$

where $n_1 = n - k_1 - k_2$ and $n_2 = n - k_k - k_{k-1}$. With $n_3 = n_1 - k_3$ and $\tilde{k} = \sum_{i=1}^{k-1} k_i$ an equivalent orthogonal basis is given by $1^{n\times 1}$ and

$$\left\{\begin{pmatrix} 1^{k_1\times 1} \\ -\dfrac{k_1}{k_2}1^{k_2\times 1} \\ 0^{n_1\times 1}\end{pmatrix}, \begin{pmatrix} 1^{(k_1+k_2)\times 1} \\ -\dfrac{(k_1+k_2)}{k_3}1^{k_3\times 1} \\ 0^{n_3\times 1}\end{pmatrix}, \ldots, \begin{pmatrix} 1^{(n-k_k)\times 1} \\ -\dfrac{\tilde{k}}{k_k}1^{k_k\times 1}\end{pmatrix}\right\}. \tag{2}$$

2.2 Matrix polynomial theory

A recent book on matrix polynomial theory is [18]. Here we summarise some of the definitions that are important for this paper. The function

$$P(\lambda) = I\lambda^2 + L_{\dot{x}}\lambda + L_x \tag{3}$$

is a quadratic matrix polynomial. The eigenvalues λ_0 of (3) are defined by det $P(\lambda_0) = 0$ and the corresponding eigenvectors by $P(\lambda_0)v = 0$, *i.e.* $(I\lambda_0^2 + L_{\dot{x}}\lambda_0 + L_x)v = 0$. Eigenproblems of quadratic matrix polynomials are genereally referred to as quadratic eigenvalue problems (QEP). For an extensive review of applications and solutions of the QEP see [19].

If L_x and $L_{\dot{x}}$ are real-valued, all the eigenvalues of $P(\lambda)$ are real or arise in complex-conjugated pairs. If L_x and $L_{\dot{x}}$ are symmetric we speak of a self-adjoint matrix polynomial. If $P(\lambda)$ is self-adjoint, then its left and right eigenvectors coincide.

Every quadratic matrix polynomial admits a number of matrix pencil linearizations, where the $n \times n$ matrix $P(\lambda)$ is transformed to a $2n \times 2n$ matrix $(P_1 - \lambda I)$, which is linear in λ and has the same eigenvalues and multiplicities as $P(\lambda)$. One of the most common linearizations involves the matrix

$$P_1 = \begin{pmatrix} 0 & I \\ -L_x & -L_{\dot{x}}\end{pmatrix} \tag{4}$$

and λ_0 is an eigenvalue of $P(\lambda)$ if and only if it is an eigenvalue of P_1.

Lemma 3 [18]**.** Let $P(\lambda)$ be a quadratic matrix polynomial and $\mathcal{L}$ the corresponding linearization with an eigenvalue λ_0. The following two statements are equivalent:

- $P(\lambda_0)$ has a right eigenvector v and left eigenvector w crorresponding to λ_0.
- $\mathcal{L}$ has a right eigenvector $(v^T, \lambda_0 v^T)^T$ and left eigenvector $(w(\lambda_0 I + L_{\dot{x}}), w)$ corresponding to λ_0.

III. MODELLING AND CONSENSUS ALGORITHM

We consider a group of n mobile agents moving in a two- or three-dimensional space. We do not specify further the considered agent type, however, we do assume that the individual agent's dynamics is decoupled along the different dimensions, *i.e.* that consensus in each direction can be investigated as a one-dimensional problem.

We denote the position of the ith agent as x^i, $i \in \{1, \ldots, n\}$ and its velocity as $\dot{x}^i$. The dynamics of an agent are goverened by $\ddot{x}^i(t) = u^i(t)$, where $u^i(t)$ is the control input. This model is fairly simple, however it reflects a number of technical applications sufficiently well.

The positions (velocities) of all agents are collected in the $n \times 1$ vector $x(\dot{x})$ and the control inputs in the vector u, thus the collected dynamics are given by

$$\ddot{x}(t) = u(t).$$

The agents move in a common reference frame and can measure their own position or velocity or both. The velocity data is then shared along a graph $G_{\dot{x}}$ and the position data along a graph G_x. The control variable $u(t)$ is determined by the following intuitive consensus algorithm for the double integrator case [5]:

$$u(t) = -\underbrace{L(G_x)}_{:=L_x}x(t) - \underbrace{L(G_{\dot{x}})}_{:=L_{\dot{x}}}\dot{x}(t).$$

The closed loop system can then be written as

$$\begin{pmatrix}\dot{x} \\ \ddot{x}\end{pmatrix} = \underbrace{\begin{pmatrix} 0^{n\times n} & I^{n\times n} \\ -L_x & -L_{\dot{x}}\end{pmatrix}}_{:=\mathcal{L}}\begin{pmatrix} x \\ \dot{x}\end{pmatrix}. \tag{5}$$

We say that an algorithm achieves velocity consensus asymptotically if for any initial condition $x_0, \dot{x}_0 \in \mathbb{R}^n$, as $t \to \infty$, $|\dot{x}^i - \dot{x}^j| \to 0$, $1 \le i, j \le n$ and that it achieves position consensus asymptotically if $|x^i - x^j| \to 0$. We will use the term consensus type to characterise what kind of consensus (position and velocity, only velocity, or none) is achieved. Consensus is bounded if the consensus value is bounded.

In [5–9] and related work $L_x = L_{\dot{x}}$ is assumed. In our work generally $L_x \neq L_{\dot{x}}$. But since G_x and $G_{\dot{x}}$ share the same set of nodes, there is a structural dependance between the two matrices: If the nodes of G_x are renamed such that L_x obtains a specific form, then the structure of $L_{\dot{x}}$ changes accordingly.

Throughout this paper we implicitly assume that if we transform one of the matrices, the other matrix changes as well.

Obviously, $\mathcal{L}$ in (5) corresponds to P_1 in (4) and the eigenvalues of $\mathcal{L}$ coincide with the eigenvalues of the quadratic matrix polynomial (3).

There is an established spectral theory for quadratic matrix polynomials in the case that L_x, $L_{\dot{x}}$ are positive definite. In particular, it is a well-known result, called law of inertia, that all eigenvalues of (3) then have positive real parts. Our starting point is that L_x, $L_{\dot{x}}$ are Laplacians of undirected graphs and therefore positive semi-definite. Due to the special form of graph Laplacians we are able to present a constructive statement on the eigenvalues of $P(\lambda)$. This extends the field of application of the QEP to consensus in multi-agent systems and the law of inertia to semi-definite Laplacian matrices.

IV. MAIN RESULT

In this section we present our main result related to undirected heterogeneous networks. First, we derive the spectrum of $\mathcal{L}$, followed by a proof of convergence.

4.1 Zero eigenvalue of $\mathcal{L}$

Let us first consider the zero eigenvalue of $\mathcal{L}$. Since $\mathcal{L}$ and $P(\lambda)$ have the same spectrum, we will use the $2n \times 2n$ matrix and the $n \times n$ quadratic matrix polynomial interchangingly.

Lemma 4. Let $\mathcal{L}$ be given by (5). Then $\lambda_0 = 0$ is an eigenvalue of $\mathcal{L}$. Furthermore $j_{\mathcal{L}}(0) = k$ if and only if $j_{L_x}(0) = k$. All the corresponding right eigenvectors are then given by $(v^T, 0^{1\times n})^T$, and all the corresponding left eigenvectors are given by $(wL_{\dot{x}}, w)$ where v is a right and w a left eigenvector of L_x corresponding to the eigenvalue 0.

Proof. The form of the eigenvectors for $\lambda_0 = 0$ follows directly from Lemma 3. The eigenproblem for $\mathcal{L}$ is then reduced to $L_x v = 0$, thus $j_{\mathcal{L}}(0) = k$ if and only if the kernel of L_x contains k linearly independent eigenvectors. □

Lemma 5. Let $\mathcal{L}$ be given by (5). It holds that $j_{\mathcal{L}}(0) = k$ with $|j_{\mathcal{L},1}(0)| = 2$, and $|j_{\mathcal{L},i}(0)| = 1$, $i = 2 \dots k$, if and only if G_x consists of k connected components and $G_x \cup G_{\dot{x}}$ is connected.

Proof. The fact $j_{\mathcal{L}}(0) = k$ stems from Lemma 4. It remains to consider the sizes of the Jordan blocks.

Let G_x have k connected components of size $k_1, \dots, k_k$. Then, according to Lemma 4, $j_{\mathcal{L}}(0) = k$. Furthermore, L_x has k linearly independent eigenvectors $v_1, \dots, v_k$ correspoinding to the eigenvalue 0, given by (1). A set of k linearly independent eigenvectors of $\mathcal{L}$ is then, according to Lemma 4, given by $u_i = (v_i^T, 0^T)^T$, $i = 1, \dots, k$.

Next, we need to show that there is one Jordan chain of length 2. Let $u_1 = (1^{1\times n}, 0^{1\times n})^T$. We see that $u_2 = (0^{1\times n}, 1^{1\times n})^T$ is a generalized eigenvector that satisfies $\mathcal{L}u_2 = u_1$, therefore $|j_{\mathcal{L},1}(0)| \geq 2$. By Lemma 1 we know that there is no vector u_3 satisfying $\mathcal{L}u_3 = u_2$. Hence $|j_{\mathcal{L},1}(0)| = 2$.

Next we show that $|j_{\mathcal{L},2}(0)| = 1$ with the eigenvector $(v_2^T, 0^{1\times n})^T$. We partition $L_x = \begin{pmatrix} L_{x1} & 0 \\ 0 & L_{x2} \end{pmatrix}$, where L_{x1} is of dimension $k_1 \times k_1$, L_{x2} of $(n - k_1) \times (n - k_1)$. If $G_x \cup G_{\dot{x}}$ is connected, we see that there must be at least one edge between the sets of nodes $\{v_1, \dots, v_{k_1}\}$ and $\{v_{k_1+1}, \dots, v_n\}$ in $G_{\dot{x}}$. Therefore the corresponding partition of $L_{\dot{x}}$ is given by $L_{\dot{x}} = \begin{pmatrix} L_{\dot{x}1} & L_{\dot{x}3} \\ L_{\dot{x}3}^T & L_{\dot{x}2} \end{pmatrix}$ where $L_{\dot{x}3} \not\equiv 0$. Clearly, if $b = (b_1^{1\times k_1}, b_2^{1\times(n-k_1)}, b_3^{1\times k_1}, b_4^{1\times(n-k_1)})^T$ is a generalized eigenvector belonging to $(v_2^T, 0^{1\times n})^T$, it must hold that $(b_3^T, b_4^T)^T = v_2 = (1^{1\times k_1}, 0^{1\times(n-k_1)})^T$ and therefore $-L_{x2}b_2 = L_{\dot{x}3}^T 1^{k_1\times 1} \neq 0$. The right hand of this equation is elementwise less than or equal to zero, therefore it follows from Lemma 1 that there is no vector b_2 that satisfies the equation. Hence, we have shown that $|j_{\mathcal{L},2}(0)| = 1$. The proof for $|j_{\mathcal{L},i}(0)| = 1$, i ($\{3, \dots, k\}$ is identical. This completes the sufficiency part of the proof.

We now show the necessity. From Lemma 4 and $j_{\mathcal{L}}(0) = k$ it follows that $j_{L_x}(0) = k$, *i.e.*, G_x consists of k connected components. To prove that $|j_{\mathcal{L},1}(0)| = 2$ and $|j_{\mathcal{L},i}(0)| = 1$, $i = 2, \dots, k$, implies connectedness of $G_x \cup G_{\dot{x}}$, assume that $G_x \cup G_{\dot{x}}$ is not connected: without loss of generality (wolog) assume that it consists of two connected components. Then we can relabel the node set V such that

$$L_x = \begin{pmatrix} L_{x1} & 0 \\ 0 & L_{x2} \end{pmatrix}, \quad L_{\dot{x}} = \begin{pmatrix} L_{\dot{x}1} & 0 \\ 0 & L_{\dot{x}2} \end{pmatrix}$$

and L_{x1}, $L_{\dot{x}1}$ (and thus L_{x2}, $L_{\dot{x}2}$) have identical dimensions. It is then seen that $\mathcal{L}$ is similar to the matrix

$$\tilde{\mathcal{L}} = \begin{pmatrix} 0 & I & 0 & 0 \\ -L_{x1} & -L_{\dot{x}1} & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -L_{x2} & -L_{\dot{x}2} \end{pmatrix} := \begin{pmatrix} \tilde{\mathcal{L}}_1 & 0 \\ 0 & \tilde{\mathcal{L}}_2 \end{pmatrix}. \tag{6}$$

Clearly, both $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ have a zero eigenvalue with a Jordan block of size 2, hence $\mathcal{L}$ has two such Jordan blocks, which contradicts $|j_{\mathcal{L},i}(0)| = 1$, $i = 2 \dots k$. □

An important special case of the above is that if G_x is connected, then $j_{\mathcal{L}}(0) = 1$ with $|j_{\mathcal{L}}(0)| = 2$. This will be relevant when analyzing the stability of the consensus algorithm.

4.2 Nonzero eigenvalues of $\mathcal{L}$

Before we give our next result, let us derive an explicit formula for the eigenvalues of $\mathcal{L}$. Multiplying $P(\lambda)v = 0$ by v^* from the left gives the quadratic equation

$$\lambda^2 v^* v + \lambda v^* L_{\dot{x}} v + v^* L_x v = 0.$$

Its coeffcients are real for all v due to the symmetry of L_x and $L_{\dot{x}}$. We can obtain λ as the solutions of

$$\lambda = \frac{-v^* L_{\dot{x}} v \pm \sqrt{(v^* L_{\dot{x}} v)^2 - 4(v^* v)(v^* L_x v)}}{2v^* v}, \tag{7}$$

where v is an eigenvector of $P(\lambda)$ [19].

Lemma 6. Let $\mathcal{L}$ be given by (5). Then $\mathcal{L}$ has no eigenvalues with positive real parts.

Proof. The matrices $L_{\dot{x}}$ and L_x are positive semi-definite, thus $\frac{v^* L_{\dot{x}} v}{v^* v} \geq 0$ and $\frac{v^* L_x v}{v^* v} \geq 0$ for all $v \neq 0$. Thus, it is evident that all solutions of (7) are real, imaginary, or complex conjugate with a nonpositive real part. □

It remains to establish when $\mathcal{L}$ has nonzero imaginary eigenvalues.

Lemma 7. Let $\mathcal{L}$ be given by (5). All eigenvalues of $\mathcal{L}$ are zero or have negative real parts if and only if for the system $\dot{\tilde{x}} = L_x \tilde{x} + L_{\dot{x}} \tilde{u}$, 0 is the only uncontrollable eigenvalue of L_x.

Proof. We have already shown that $\mathcal{L}$ has no eigenvalues with a positive real part. We see that (7) has the imaginary solution $\lambda = \pm\gamma i$, $\gamma \in \mathbb{R}^+$, if and only if

$$\exists v \neq 0: \quad v \in \ker(L_{\dot{x}}) \quad \text{and} \quad L_x v = \gamma^2 v. \tag{8}$$

Whenever such a v exists, $P(\gamma i)v = 0$ is reduced to $(L_x - I\gamma^2)v = 0$. As L_x and $L_{\dot{x}}$ are symmetric, we can write the conditions (8) in matrix form as

$$\exists v \neq 0: \quad v^T(\gamma^2 I - L_x \mid L_{\dot{x}}) = 0, \text{ i.e.}$$

$$\operatorname{rank}(\gamma^2 I - L_x \mid L_{\dot{x}}) < n.$$

Using the well-known Popov–Belevitch–Hautus test [20], this is equivalent to $\gamma^2 > 0$ being an eigenvalue of L_x which is uncontrollable for the pair $(L_x, L_{\dot{x}})$. □

Corollary 1. Let $\mathcal{L}$ be given by (5) and $G_{\dot{x}}$ be connected. Then $\mathcal{L}$ has no imaginary eigenvalues.

Proof. If $G_{\dot{x}}$ is connected, then the only vector in the kernel of $L_{\dot{x}}$ is $1^{n\times 1}$, which is also in the kernel of L_x. The result follows from Lemma (7). □

4.3 Convergence

We have shown that $\mathcal{L}$ has at least one Jordan block of size 2 corresponding to the zero eigenvalue, i.e. (5) is not stable in the classical sense. However, stability is not the required system behaviour. Clearly, if the agents achieve velocity consensus and agree on some constant velocity, their positions will still evolve in time. We are now ready to present our main result.

Theorem 1. Consider the double integrator consensus problem for n mobile agents. Let the position information be shared along the communication network G_x and the velocity information along $G_{\dot{x}}$. Algorithm (5) achieves velocity consensus asymptotically if and only if

(i) $G_x \cup G_{\dot{x}}$ is connected,
(ii) $\operatorname{rank}(\lambda I - L_x \mid L_{\dot{x}}) = n \quad \forall \lambda \in \mathbb{C} \setminus \{0\}$.

It achieves velocity and position consensus asymptotically if and only if (i–ii) hold and additionally

(iii) G_x is connected.

Proof. The second part of this theorem is equivalent to $j_{\mathcal{L}}(0) = 1$, $|j_{\mathcal{L}}(0)| = 2$ and all the other eigenvalues having negative real parts, which is a special case of [5] and the proof is given there. For the first part, note that with Lemma 5, Lemma 6 and Lemma 7, (i–ii) ensures that $\mathcal{L}$ has no purely imaginary eigenvalues or eigenvalues with postive real parts, as well as $j_{\mathcal{L}}(0) = k$, $|j_{\mathcal{L},1}(0)| = 2$, $|j_{\mathcal{L},i}(0)| = 1$, $i = 2, \ldots, k$, where k is the number of connected components of G_x.

The Jordan canonical form of $\mathcal{L}$, $\mathcal{J}(\mathcal{L})$ is given by

$$V^{-1}\mathcal{L}V = \begin{pmatrix} w_0 \\ \vdots \\ w_{2n-1} \end{pmatrix} \mathcal{L}(u_0 \quad \ldots \quad u_{2n-1}) = \mathcal{J},$$

where u_i can be chosen among the right eigenvectors and generalized eigenvectors of $\mathcal{L}$, and w_i are left eigenvectors and generalized eigenvectors of $\mathcal{L}$ scaled and allocated accordingly. Let L_x have k connected components of size $k_1, \ldots, k_k$. We can choose $u_0 = (1^{1\times n}, 0^{n\times 1})^T$, $u_1 = (0^{1\times n}, 1^{1\times n})^T$ and thus $w_0 = (p_1 1^{1\times n}, 0^{1\times n})$, $w_1 = (0^{1\times n}, p_1 1^{1\times n})$, $p_1 = 1/n$. We know that L_x admits the additional set of eigenvectors (2), denoted $v_2, \ldots, v_k$. We then know from Lemma 4 that $u_i = (v_i^T, 0^{1\times n})^T$ can be chosen as the right eigenvectors of $\mathcal{L}$ associated with the zero eigenvalue. The Jordan matrix then has the form

$$\mathcal{J} = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & \tilde{J} \end{pmatrix},$$

where $J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $J = 0^{(k-1)\times(k-1)}$ are the collected Jordan blocks corresponding to the zero eigenvalue and $\tilde{J}$ are the remaining Jordan blocks. Thus

$$e^{\mathcal{L}t} = e^{V\mathcal{J}tV^{-1}} = V\begin{pmatrix} e^{J_0 t} & 0 & 0 \\ 0 & e^{Jt} & 0 \\ 0 & 0 & e^{\tilde{J}t} \end{pmatrix} V^{-1}. \quad (9)$$

Since all nonzero eigenvalues of $\mathcal{L}$ have negative real parts we know that $e^{\tilde{J}t} \to 0$ as $t \to \infty$. On the other hand $e^{J_0 t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $e^{Jt} = I^{(k-1)\times(k-1)}$ Denote by $w_2, \dots, w_k$ the lines $2 \dots k$ of V^{-1}. Then for $t \to \infty$

$$e^{\mathcal{L}t} \to (u_0 \quad u_1)\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + (u_2 \dots u_k)\begin{pmatrix} w_2 \\ \vdots \\ w_k \end{pmatrix}$$

or equivalently

$$e^{\mathcal{L}t} \to \frac{1}{n}\begin{pmatrix} 1^{n\times1} & 0^{n\times1} \\ 0^{n\times1} & 1^{n\times1} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1^{1\times n} & 0^{1\times n} \\ 0^{1\times n} & 1^{1\times n} \end{pmatrix} + \begin{pmatrix} v_2 & \dots & v_k \\ 0^{n\times1} & \dots & 0^{n\times1} \end{pmatrix}\begin{pmatrix} w_2 \\ \vdots \\ w_k \end{pmatrix}. \quad (10)$$

Thus we obtain

$$\lim_{t\to\infty} \dot{x}(t) = \left(\frac{1}{n}\sum_{j=1}^{n} \dot{x}^j(0)\right)1^{n\times1}$$

which is bounded velocity consensus.

For necessity, note that if condition (i) is violated, the resulting Jordan matrix has at least two blocks of the form J_0 and thus no global convergence is achieved. If condition (ii) is violated, then there is at least one imaginary eigenvalue pair $\pm i\gamma$ and $\mathcal{J}$ contains a block $\begin{pmatrix} i\gamma & 0 \\ 0 & -i\gamma \end{pmatrix}$. The corresponding left eigenvector is $(v^T, \pm i\gamma v^T)^T$, where v lies in the nullspace of $L_{\dot{x}}$ and is not $1^{n\times1}$. Thus the agents' velocities oscillate without converging. □

Note that if G_x has k connected components, then the individual components will achieve position consensus within themselves with a constant offset between the agent groups. This result is physically plausible: while position consensus is impossible without velocity consensus, the converse makes sense. With $x(t) = \int \dot{x}(\tau)d\tau + d_x$ we see that d_x is exactly the offset produced by the different connected components. This offset can be calculated from (10) as

$$d_x = \left(\frac{1}{n}\sum_{i=1}^{n} x(0)\right)1^{n\times1} + (v_2 \dots v_k)\begin{pmatrix} w_2 \\ \vdots \\ w_k \end{pmatrix}\begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix}. \quad (11)$$

Condition (ii) in Theorem 1 ensures that $\mathcal{L}$ has no imaginary eigenvalues. One of its direct consequences is that agents with double integrator dynamics cannot achieve consensus using only position information. Indeed, if $L_{\dot{x}} \equiv 0$ then all nonzero eigenvalues of $\mathcal{L}$ are imaginary.

In general, checking condition (ii) is equivalent to checking if there is an eigenvector of L_x that lies in the kernel of $L_{\dot{x}}$, *i.e.*, that is a linear combination of the vectors in (1). Thus the problem is reduced to finding necessary and sufficent conditions for eigenvectors of L_x to have repeated entries. To our best knowledge, there are no straightforward, graph-based necessary conditions that state the shape of the eigenvectors of a Laplacian, except for some special cases. However, several sufficient conditions can be found, so that the presence of imaginary eigenvalues can often be seen from the structure of G_x. We list here the most interesting cases. For the rest of this section we assume wolog that condition (i) holds and that $G_{\dot{x}}$ has at least two connected components.

Lemma 8. Let $G_{\dot{x}}$ consist of k connected components $V_1, \dots, V_k$, where $k < n$. If G_x has an almost equitable m-partition $W_1, \dots, W_m$ $m \le k$, such that $V_1, \dots, V_k$ is at least as fine as $W_1, \dots, W_m$, then $\mathcal{L}$ has imaginary eigenvalues. $(\forall (I_i)_{1\le i\le m} \subseteq \{1,\dots,k\} : W_i = \bigcup_{j\in I_i} V_j) \vee (\bigcup_{1\le i\le m} W_i = V) \vee (\forall i \neq j : W_i \bigcap W_j = \emptyset)$, here V is the node set of G_x and $G_{\dot{x}}$.

Proof. Let G_x have an almost equitable partition $W_1, \dots, W_m$ of size $m_1, \dots, m_m$. With $V_1, \dots, V_k$ of size $k_1, \dots, k_k$ as above, any linear combination of the vectors in (1) is in the kernel of $L_{\dot{x}}$. By [17], Proposition 2, L_x has an eigenvector $v = (\underbrace{\beta_1 \dots \beta_1}_{m_1}, \underbrace{\beta_2 \dots \beta_2}_{m_2}, \dots, \underbrace{\beta_m \dots \beta_m}_{m_m})$, $\beta_i \in \mathbb{R}$ with a corresponding positive eigenvalue. Thus v can be chosen as a linear combination of the vectors in (1) and the condition (8) from Lemma 7 is satisfied.

Lemma 9. Let $G_{\dot{x}}$ consist of two connected components on node sets V_1, V_2. Then $\mathcal{L}$ has imaginary eigenvalues if and only if (V_1, V_2) is an almost equitable 2-partition of G_x.

Proof. Sufficiency follows from Lemma 8. To show necessity, note that for an almost equitable 2-partition with the characteristic matrix Q_x [17] $L_x Q_x b = \gamma^2 b$, $\gamma \in \mathbb{R}^+$, where $b = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $Q_x = \begin{pmatrix} 1^{k_1\times1} & 0^{k_1\times1} \\ 0^{k_2\times1} & 1^{k_2\times1} \end{pmatrix}$. If the 2-partition is almost equitable, each node in V_1 (V_2) has d_{12} (d_{21}) neighbors in V_2 (V_1), *i.e.* $L_x Q_x = \begin{pmatrix} d_{12}^{k_1\times1} & -d_{12}^{k_1\times1} \\ -d_{21}^{k_2\times1} & d_{21}^{k_2\times1} \end{pmatrix}$. Let the partition be not almost equitable, wolog let there be one node $v_{k_1} \in V_1$ such that it has $e_{12} \neq d_{12}$ neighbors in V_2. Then

$$L_xQ_xb = \begin{pmatrix} d_{12}^{(k_1-1)\times 1} & -d_{12}^{(k_1-1)\times 1} \\ e_{12} & -e_{12} \\ -d_{21}^{k_2\times 1} & d_{21}^{k_2\times 1} \end{pmatrix} b \overset{!}{=} \gamma^2 Q_x \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

Then $d_{12}(\beta_1 - \beta_2) = \gamma^2\beta_1$ and $e_{12}(\beta_1 - \beta_2) = \gamma^2\beta_1$ must hold simultaneously. Thus either $\gamma^2 = 0$ or $d_{12} = e_{12}$ and the partition is almost equitable. □

Particularly, if $G_{\dot{x}}$ has one isolated node v and the remaining graph is connected, then $\mathcal{L}$ has imaginary eigenvalues if and only if $d(v) = n-1$ in G_x, *i.e.* if the isolated node is connected to all nodes of G_x.

Lemma 10. If G_x is fully connected then $\mathcal{L}$ has imaginary eigenvalues for any disconnected $G_{\dot{x}}$.

Proof. If G_x is fully connected then its Laplacian is given by $L_c = nI - 1^{n\times n}$. Choose v from (2), *i.e.* $v \in \ker(L_{\dot{x}})$ and $v^T 1^{n\times 1} = 0$. Then $L_c v = nv - 1^{n\times 1}v = nv$ and thus condition (8) is always satisfied. □

Further, more complex conditions can be found based on graph automorphism groups and other properties of the Laplacian. However to date, no full set of conditions for a Laplacian to have repeated eigenvector entries has been found by the authors.

V. CONVERGENCE RATE

We expect that the convergence rate of (5) largely depends on the chosen communcation topologies. It is a known result that the convergence rate of the single integrator consensus is bounded by the second smallest eigenvalue of the corresponding Laplacian matrix. In this section we expand the result for the algorithm (5). In the following we assume that conditions (i)–(ii) of Theorem 1 hold.

Define the group position error vector as $e(t) = x(t) - d_{\dot{x}}t - d_x$. Here $d_{\dot{x}} = \left(\frac{1}{n}\sum_{i=1}^{n} \dot{x}^i(0)\right)1^{n\times 1}$ is the vector average velocity at $t = 0$ and d_x, given by (11), is the position offset after velocity consensus has been achieved. Note that $d_{\dot{x}}$ lies in the kernel of any Laplacian matrix and d_x lies in the kernel of L_x. The error dynamics of the second order consensus algorithm are then given by

$$\ddot{e} = \frac{d}{dt}(-L_x x - L_{\dot{x}}\dot{x}) = -L_x e - L_{\dot{x}}\dot{e}$$

or, rewritten as a first order system,

$$\begin{pmatrix} \dot{e} \\ \ddot{e} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L_x & -L_{\dot{x}} \end{pmatrix}\begin{pmatrix} e \\ \dot{e} \end{pmatrix}. \tag{12}$$

Let $\lambda_{crit} = \min_{i\in\{1,\ldots,2n\},\lambda_i\neq 0} |\mathrm{Re}(\lambda_i)|$, where λ_i are the eigenvalues of $\mathcal{L}$. Consider $e^{\mathcal{L}t}$ in (9). The block $e^{\tilde{\mathcal{J}}t}$ converges to zero with a rate that is equal to or faster than λ_{crit} and therefore (12) tends to zero with a rate that is equal to or faster than λ_{crit}. In order to find the value of λ_{crit} we use the numerical range

$$0 \le \frac{v^*Lv}{v^*v} \le \mu_n, \quad v \not\equiv 0$$

where μ_n is the largest eigenvalue of the Laplacian. We are interested in the solutions of (7), thus only such v are considered which are eigenvectors of $P(\lambda)$ corresponding to an eigenvalue with a nonzero real part.

Let a graph have k connected components of size $k_1, \ldots, k_k$, and G_{k_i} be the graph corresponding to the connected component of size k_i. Then the largest eigenvalue of the corresponding Laplacian is bounded by [21] $\min_{i,k_i\neq 1} \frac{k_i}{k_i - 1}\max_j d(v_j \in G_{k_i}) \le \mu_n \le \max_i k_i$. Here $d(v_j)$ denotes the degree of node v_j. Note that $\max_j d(v_j \in G_{k_i}) \le k_i - 1$.

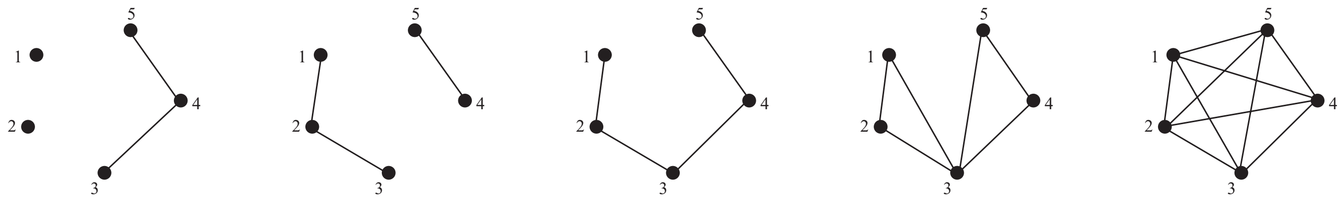
Let $G_{\dot{x}}$ have $k^{\dot{x}}$ connected components of size $k_1^{\dot{x}}, \ldots, k_{k^{\dot{x}}}^{\dot{x}}$, whith $k_{max}^{\dot{x}} = \max_i k_i^{\dot{x}}$. Let G_x have k^x connected components, with, analogously, $k_{max}^x = \max_i k_i^x$ and assume that there are no purely imaginary eigenvalues. Then $0 < \frac{v^*L_{\dot{x}}v}{v^*v} \le k_{max}^{\dot{x}}$.

Choose $v^*v = 1$ and consider (7). We see that if $(v^*L_{\dot{x}}v)^2 \le 4v^*L_xv$, λ_{crit} depends entirely on the eigenvalues of $L_{\dot{x}}$, particularly $\lambda_{crit} \le k_{max}^{\dot{x}}$. Thus, if $k_{max}^{\dot{x}} \ll k_{max}^x$, it is possible that $\mathcal{L}$ will have eigenvalues where the real and the imaginary part have a small resp. large magnitude. On the other hand, if $L_{\dot{x}}$ is "well" connected and k_{max}^x is small, the eigenvalues of $\mathcal{L}$ will be real or have small imaginary parts. This leads us to the surprising conclusion that if only a small number of agents can exchange their velocity (*i.e.*, $k_{max}^{\dot{x}}$ is small), the number of agents exchanging their position should be as small as possible, too, in order to avoid oscillations. But this in turn leads to a small λ_{crit}. We further see that λ_{crit} is bounded by

$$0 < \lambda_{crit} \le n,$$

where the upper bound is tight (choose $G_{\dot{x}}$ fully connected and G_x empty).

Example 1. Consider the graphs in Fig. 1. Let G_x be graph 1, $G_{\dot{x}}$ be graph 2. Both graphs are disconnected, however $G_x \cup G_{\dot{x}}$ is connected and {1,2,3}, {4,5} is not an almost equitable partition of G_x. As G_x has three connected components, we expect the system to achieve velocity consensus and the agents 3–5 to achieve position consensus.



(a) Graph 1 (b) Graph 2 (c) Graph 3 (d) Graph 4 (e) Graph 5

Fig. 1. Different communication topologies for five agents.

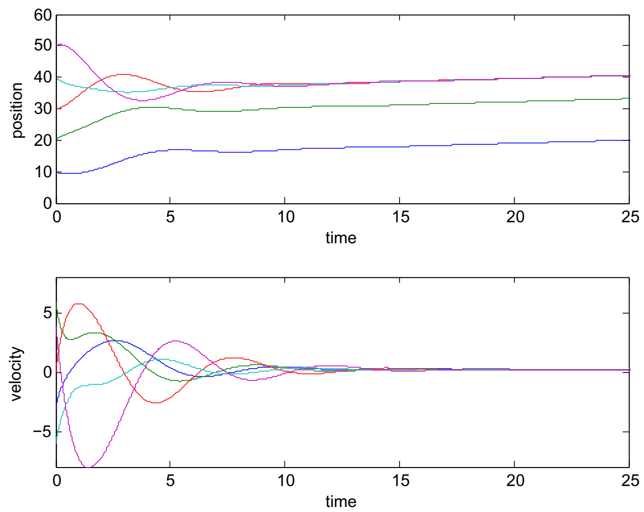


Fig. 2. G_x as graph 1, $G_{\dot{x}}$ as graph 2, velocity consensus.

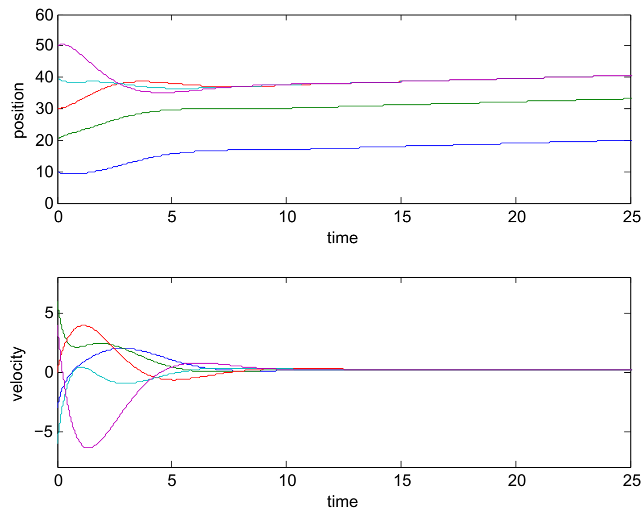


Fig. 3. G_x as graph 1, $G_{\dot{x}}$ as graph 4, velocity consensus.

This is validated by the simulation in Fig. 2 with initial positions from the interval (0, 50). We see that the agents continue to move at a fixed distance. The spectrum of $\mathcal{L}$ is spec($\mathcal{L}$) = {0, 0, 0, 0, –2.96, –0.84 ± 1.26*i*, –0.30 ± 0.94*i*, –0.77}, which explains the high amount of oscillation in the velocity plot. Here $k_{max}^{x} = k_{max}^{\dot{x}} = 3$ coincides with the largest eigenvalue of both Laplacians.

Choosing $G_{\dot{x}}$ as graph 4 in Fig. 1 leads to $k_{max}^{\dot{x}} = 5$ and the numerical range is now given by $0 \le \frac{v^* L_{\dot{x}} v}{v^* v} \le 3.6$. As can be seen in Fig. 3 consensus is achieved faster and without oscillations.

Example 2. Let G_x be given by graph 3 in Fig. 1. Choosing $G_{\dot{x}}$ as the complete graph (graph 5 in Fig. 1) leads to $\mathcal{L}$ having only real eigenvalues. The system achieves velocity and position consensus asymptotically. The simulation results are given in Fig. 4.

On the other hand, choosing G_x as a fully connected graph and $G_{\dot{x}}$ as graph 3 leads to $\mathcal{L}$ having eigenvalues with large imaginary parts and real parts that are so small that they are numerically rounded to zero. Here $\frac{v^* L_{\dot{x}} v}{v^* v} \le 3.6$ and $\frac{v^* L_x v}{v^* v} \le 5$.

VI. OUTLOOK

In the present work we have studied consensus over constant, undirected networks. While this assumption holds for many technical applications, it is certainly only a first step towards a complete understanding of double integrator consensus. Directed networks need to be considered if the information is assumed to be broadcasted by the agents, while the communication topology itself should be modelled as switching if, *e.g.*, sensor breakdowns are to be accounted for. Our current work focuses on expanding the results in this paper to the case of directed graphs.

Connectivity is defined differently for directed graphs. We say that a graph contains a spanning tree if there is a node v_i sucht that there is a directed path from v_i to any other node in the graph. It is a classic result that single integrator agents reach consensus over a network containing a spanning tree.

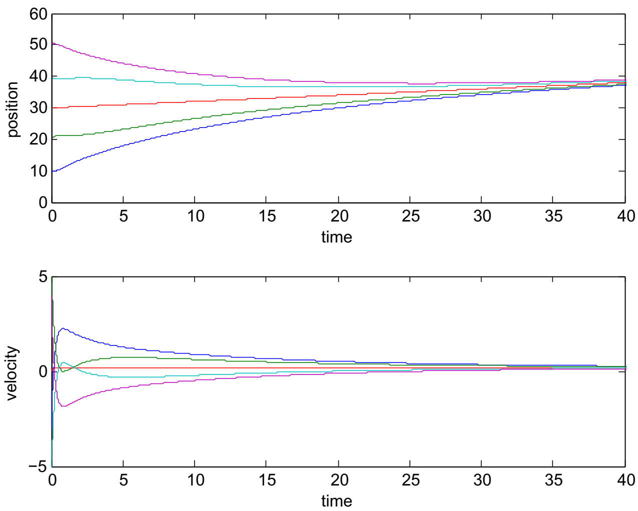


Fig. 4. G_x as graph 3, $G_{\dot{x}}$ as a fully connected graph, velocity and position consensus.

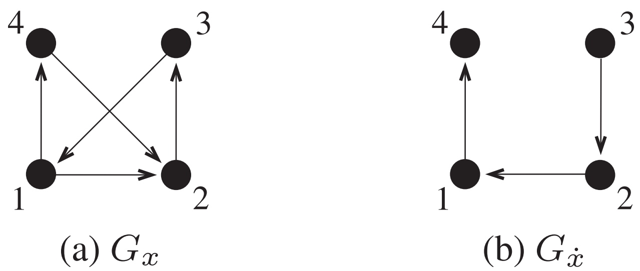


Fig. 5. Two directed communication topologies.

However, this result does not translate to the double integrator case. Consider Fig. 5. Here, G_x and $G_{\dot{x}}$ both contain a spanning tree, but $\mathcal{L}$ has positive eigenvalues and the algorithm (5) does not achieve consensus. Furthermore, choosing $\ddot{x} = -L_{\dot{x}}\dot{x}$, *i.e.*, assuming G_x to be empty, achieves velocity consensus. Thus, additional edges in G_x may lead to instability when directed graphs are considered. This is also the case for homogeneous networks, cf. [5]. Moreover we can find examples where $G_{\dot{x}}$ does not contain a spanning tree, but the algorithm achieves consensus.

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Darina Goldin is currently working toward Ph.D. degree in Control Systems Group of Technische Universitaet (TU) Berlin (Germany). She has studied mechanical and physical engineering at TU Braunschweig (Germany), TU Berlin and Università degli Studi La Sapienza (Italy) and received her Diploma degree in 2009. Her research interests include multiagent systems, complex networks and systems and graph theory.

Jörg Raisch is professor at TU Berlin, where he heads Control Systems Group within Department of Electrical Engineering and Computer Science. He is also head of Systems and Control Theory Group at Max Planck Institute for Dynamics of Complex Technical Systems in Magdeburg (Germany). His research interests are in hybrid systems and hierarchical control and include biomedical control and chemical process control applications.