Modeling and Control of Nested Manufacturing Processes using Dioid Models

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***Abstract*—Manufacturing systems are frequently adapted to changing customer demands. However, every extension with respect to the hardware requires a modification of the corresponding model and controller. In this paper we propose a dioid model of manufacturing processes in which parts may visit the same resource more than once. The proposed model can be used to determine a controller that maintains the throughput while starting activities just-in-time. Furthermore, the model can easily be adapted in case of hardware modifications. Also nested processes, i.e., processes in which some activities of part k may be executed prior to activities of part $(k-1)$, can be modeled with the proposed approach.**

I. Introduction

Fully automated manufacturing systems are used to produce large numbers of identical parts. Often such production facilities are manufacturing lines, where every resource performs at most one processing step on every part. In some manufacturing processes, however, parts may visit the same resource more than once. An example, where parts may revert to a resource they have already visited, are high-throughput screening (HTS) plants [1]. Such HTS plants are used to quickly test thousands of biochemical compounds. Thus, HTS systems are not really considered to be manufacturing systems, however, they have the same characteristics as the considered manufacturing processes.

For the case that a resource may perform more than one processing step on the same part, the operation of the plant may be more complex. This is due to the fact that the optimal manufacturing schedules may be nested. Nested schedules are schedules where activities performed on part k may be scheduled prior to activities which belong to the previous part, e.g., part $(k-1)$. Such schedules are also called overtaking or interleaving [2].

A method to determine globally optimal schedules for nested manufacturing processes, such as HTS systems, has been introduced in [3]. This approach is based on a cyclic timed discrete-event models. The cyclic character is a consequence of producing identical parts. Timed discrete-event systems are in general quite convenient for modeling production processes. Such systems are driven by the occurrence of different events, e.g., the start/end event of a sub-process or activity, and the timing information includes minimal durations between the occurrence of these events.

Due to increasing customer demands, production systems may be extended in terms of hardware, e.g., adding a supplementary resource. In such a case, however, the optimal schedule changes and therefore also the corresponding model and controller have to be adapted.

In this paper, a model for cyclic (and nested) manufacturing processes in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is proposed. If the hardware of the system is changed, e.g., an additional resource is available, and the throughput of the system can be increased by running the plant with a different schedule the model can easily be adapted. Furthermore, the model can be used to determine a feedback controller which guarantees to maintain the optimal throughput while starting all activities just-in-time. Another advantage of using the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is that the obtained controller can easily be rewritten in a form which can be implemented on a PLC. This paper is based on previous publications of the authors [1], [4]. Differences are extensions in the modeling approach of the manufacturing systems.

This paper is structured as follows. Section II briefly describes the mathematical theories used in our approach. The model containing the process to manufacture a single part and the processes cyclic behavior is introduced in Section III. A small example is given to illustrate the modeling process. Section IV describes how the obtained model can be used to determine a state feedback controller. In Section V the model of the running example is adapted for a possible hardware extension of the process. Conclusions are given in Section VI.

II. Mathematical Background

A. Dioids

A dioid is an idempotent semiring, i.e., an algebraic structure $(\mathcal{D},\oplus,\otimes)$ containing two binary operations, addition $(\oplus)$ and multiplication $(\otimes)$, defined on $\mathcal{D}$ [5], where $(\mathcal{D},\oplus)$ and $(\mathcal{D},\otimes)$ each constitute a semigroup. While addition is associative, commutative and idempotent, i.e., $a\oplus a = a \forall a \in \mathcal{D}$, multiplication is associative but not necessarily commutative. Multiplication is left- and right-distributive with respect to $\oplus$. The neutral elements of $\oplus$ and $\otimes$ are denoted ε and e, respectively.

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Similar to standard algebra, the addition and multiplication defined for scalars can be extended to matrix dioids. For matrices $A, B \in \mathcal{D}^{n\times m}$ and $C \in \mathcal{D}^{m\times l}$ addition and multiplication are defined by

$$[A \oplus B]_{ij} = [A]_{ij} \oplus [B]_{ij}$$
$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{m} \left([A]_{ik} \otimes [C]_{kj}\right).$$

Powers in dioids are defined by $a^i = a \otimes a^{i-1}$ with $a^0 = e$. Often the multiplication sign is omitted in written equations when they are unambiguous.

Due to the idempotency property, any dioid can be endowed with a natural (partial) order defined by $a \preceq b \Leftrightarrow a \oplus b = b$, i.e., the sum of two elements a and b is the least upper bound of a and b.

An idempotent semiring is said to be complete if it is closed for infinite sums, i.e., there exists the greatest element of $\mathcal{D}$ given by $\top = \bigoplus_{x\in\mathcal{D}} x$, and if $\otimes$ distributes over infinite sums. The implicit equation $x = ax \oplus b$ defined over a complete idempotent semiring $(\mathcal{D}, \oplus, \otimes)$ admits $x = a^*b$, with $a^* := \bigoplus_{i=\mathbb{N}_0} a^i$, as the least solution. Some useful properties of the star operator in dioids include the following:

$$a^* (ba^*)^* = (a \oplus b)^* = (a^*b)^* a^*$$
$$(ab)^* a = a (ba)^* \tag{1}$$

One of the most commonly used (and probably the best studied) idempotent semiring is the $(\max, +)$-algebra. In this dioid the addition is defined by the standard max operator, i.e., $a \oplus b := \max(a, b)$, and multiplication of two elements is defined by the standard addition of these two elements, i.e., $a \otimes b := a + b$. These operations are defined on the set $\mathbb{Z} \cup \{-\infty\}$ and the neutral elements for addition and multiplication are $\varepsilon = -\infty$ and $e = 0$, respectively. The resulting dioid $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes)$ does not constitute a complete dioid. However, if one adds the element $\top = +\infty$ (also called top element) to the set, the resulting dioid $\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{\pm\infty\}, \oplus, \otimes)$ is complete.

Similarly, the $(\min, +)$-algebra, the dual dioid to $(\max, +)$-algebra, is defined on the set $\mathbb{Z} \cup \{+\infty\}$. In this dioid addition is defined by the standard minimization, and multiplication is the standard addition, i.e., $a \oplus' b := \min(a, b)$ and $a \otimes' b = a+b$. The corresponding neutral elements of $(\min, +)$-algebra are $\varepsilon' = +\infty$ and $e' = 0$. As natural (partial) order is again defined by $a \preceq b \Leftrightarrow a \oplus' b = b$, $\top' = -\infty$ is the top element of the complete dioid $\overline{\mathbb{Z}}_{\min} = (\mathbb{Z} \cup \{\pm\infty\}, \oplus', \otimes')$.

Note that, in general, the zero element of a dioid is the top element of the corresponding dual dioid (and vice versa), while the unit element e is the same in both dioids, i.e.,

$$\varepsilon = \top'; \quad \top = \varepsilon'; \quad e = e'.$$

Furthermore, if a dioid $\mathcal{C}$ is a semifield, i.e., each element admits a multiplicative inverse, the dual multiplication $\odot$ of two scalar elements $a, b \in \mathcal{C}$ is defined as the standard multiplication in the corresponding dual dioid $\mathcal{C}'$, i.e., $a \odot b = a \otimes' b$. Note that, $\varepsilon \odot \top = \varepsilon \otimes' \top = \top' \otimes' \varepsilon' = \varepsilon' = \top$. For two matrices $A \in \mathcal{C}^{p\times n}$ and $B \in \mathcal{C}^{n\times q}$ the dual product is defined as

$$[A \odot B]_{ij} = \bigwedge_{k=1}^{n} \left([A]_{ik} \odot [B]_{kj}\right),$$

where $\wedge$ represents the greatest lower bound.

Let us now consider $\mathbb{B}[\![\gamma, \delta]\!]$, the set of formal power series in two variables (γ, δ) with Boolean coefficients, i.e., $\mathbb{B} = \{\varepsilon, e\}$, and with exponents in $\mathbb{Z}$, i.e, $\mathbb{B}[\![\gamma, \delta]\!] = \left\{\bigoplus_{i,j\in\mathbb{Z}} \beta_{ij}\gamma^i\delta^j | \beta_{ij} \in \mathbb{B}\right\}$. Two elements $x, y \in \mathbb{B}[\![\gamma, \delta]\!]$ are equivalent, $x\mathcal{R}y$, iff $\gamma^* \left(\delta^{-1}\right)^* x = \gamma^* \left(\delta^{-1}\right)^* y$. The quotient dioid of $\mathbb{B}[\![\gamma, \delta]\!]$ with respect to $\mathcal{R}$ is denoted $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. In addition to the conventional sum and product of series, the following rules apply for the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ [5]:

$$\gamma^k\delta^t \oplus \gamma^l\delta^t = \gamma^{\min(k,l)}\delta^t$$
$$\gamma^k\delta^t \oplus \gamma^k\delta^\tau = \gamma^k\delta^{\max(t,\tau)}$$
$$\gamma^k\delta^t \otimes \gamma^l\delta^\tau = \gamma^{(k+l)}\delta^{(t+\tau)}.$$

The neutral elements for addition and multiplication of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ are $\varepsilon = \gamma^{+\infty}\delta^{-\infty}$ and $e = \gamma^0\delta^0$. The dioid is complete with top element $\top = \gamma^{-\infty}\delta^{+\infty}$. In the sequel, the dual multiplication $\odot$ of monomials in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is considered, with the following rules: $\gamma^k\delta^t \odot \gamma^l\delta^\tau = \gamma^{(k+l)}\delta^{(t+\tau)}$. It is important to note that in this dioid the dual multiplication for two monomials a, b, with $\varepsilon \prec a, b \prec \top$, is identical to the standard multiplication of these two monomials.

This algebraic structure is very efficient to model timed event graphs, a subclass of timed Petri-nets which are choice-free but include synchronization phenomena. More precisely, the dynamical behavior of a timed event graph can be described by a linear model in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ [5]. In terms of timed event graphs, the monomial $\gamma^k\delta^t \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ may be interpreted as: *the k'th event occurs at the latest at time t* or *at time t at least k events have occurred* [6].

B. Periodic Behavior in Discrete-Event Systems

Naturally, cyclic discrete-event systems evolve in a periodic manner. Periodic series in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ are usually represented in the form $s = p \oplus q \otimes r^*$. In such series, the term p is a polynomial referring to a transient phase, e.g., the start-up of the system. The term q is then a polynomial which represents the periodical behavior, i.e., the pattern which will be repeated with a periodicity given by $r = \gamma^\nu\delta^\tau$. Then the ratio ν/τ is the throughput of the series, i.e., an event occurs ν times every τ time units, once the periodic regime is reached.

For example, consider the series $s \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$:

$$s = \underbrace{e \oplus \gamma\delta^2 \oplus \gamma^3\delta^3 \oplus \gamma^4\delta^5}_{p} \oplus \underbrace{\left(\gamma^5\delta^6 \oplus \gamma^6\delta^8\right)}_{q} \underbrace{\left(\gamma^3\delta^3\right)^*}_{r^*}$$

The graphical representation of this series is given in Fig. 1. Note that in $\mathbb{B}[\![\gamma, \delta]\!]$, the monomial $\gamma^k\delta^\tau$ is represented as the point (k, τ) in the event-time-domain. In $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ this monomial is identified with all points in the south-east cone of (k, τ) (striped area in Fig. 1).

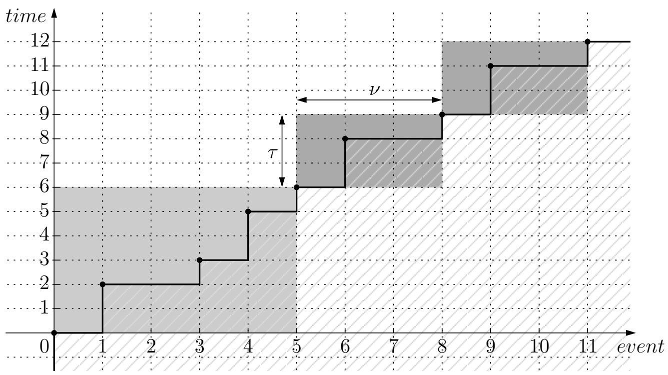

Fig. 1. Graphical representation of the series $s = e \oplus \gamma\delta^2 \oplus \gamma^3\delta^3 \oplus \gamma^4\delta^5 \oplus (\gamma^5\delta^6 \oplus \gamma^6\delta^8)\,(\gamma^3\delta^3)^*$.

C. Residuation Theory

In general dioids, an inverse is not defined. Residuation theory, however, allows defining *pseudo-inverses* of some isotone maps on ordered sets and, in particular, on dioids [5]. An isotone, or order preserving, mapping f from one partially ordered set $\mathcal{S}$ to another partially ordered set $\mathcal{T}$ is a mapping that satisfies $a \preceq b \Rightarrow f(a) \preceq f(b), \forall a, b \in \mathcal{S}$. It is called residuated if there exists a unique mapping $f^\sharp : \mathcal{T} \to \mathcal{S}$ such that $f \circ f^\sharp \succeq \mathrm{Id}_\mathcal{T}$ and $f^\sharp \circ f \succeq \mathrm{Id}_\mathcal{S}$, where $\mathrm{Id}_\mathcal{T}$ and $\mathrm{Id}_\mathcal{S}$ refer to the identity functions on $\mathcal{T}$ and $\mathcal{S}$, respectively [7]. If f is residuated, $f^\sharp$ is called the residual of f. It can be shown that $f^\sharp(y)$ is the least upper bound of the subset $\{x \,|\, f(x) \preceq y\}$.

Among the most frequently used isotone mappings are the left and right multiplication over a complete dioid, i.e., $L_a : x \mapsto a \otimes x$ and $R_a : x \mapsto x \otimes a$. Both mappings are residuated and their residuals are denoted $L_a^\sharp(x) = a \backslash\!\!\circ\, x$ and $R_a^\sharp(x) = x \,/\!\!\circ\, a$, respectively.

III. Model of Manufacturing Systems

In manufacturing systems or similar fully automated systems the user defines the specific operations he or she wants to run. More specifically, for each part the system shall produce a certain number of μ worksteps or activities are defined. These μ activities are executed on ϱ resources. Each activity i is therefore assigned to a specific resource $J(i) \in \{R_1, \ldots, R_\varrho\}$. During the execution of activity i, resource $J(i)$ is said to be occupied.

A. Model of Processing a Single Part

To illustrate our modeling and control approach we introduce a small example. It can be seen as a part of a production process composed of three activities. These activities are executed on two different resources. Note, that we only consider manufacturing processes in which parts may visit the same resource more than once, i.e., in general the total number of activities necessary to process one part is larger than the number of resources in the system ($\mu > \varrho$).

In our example, the first activity act_1, executed on resource R_1, can be seen as a preparation process for the second activity (act_2), which is the main process, executed on R_2. After finishing the second activity the part is returned to the first resource R_1 for a post-processing activity act_3. In general, two resources may be occupied simultaneously by a single part. This is, for example, the case during the transfer of a part from one resource to another.

Besides the specific sequence in which a part shall be processed, the user defines minimal times for the duration of every activity. This can easily be done by means of timed event graphs (TEGs). Timed event graphs are a class of Petri nets, in which every place has exactly one upstream and one downstream transition. Due to this characteristic they are especially suitable for modeling synchronization phenomena as they frequently appear in manufacturing processes.

The timed event graph of our running example is shown in Fig. 2. The dashed boxes in Fig. 2 indicate the three different

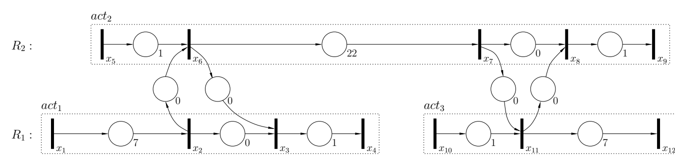

Fig. 2. Timed event graph for the production of one part.

activities. A single activity may contain a pre-processing component (e.g., warm-up), a processing component, a post-processing component (e.g., cool-down), and multiple transfer components. During a part's transfer from one resource to another resource, the corresponding transitions have to be synchronized. Activity act_2, for example, consists of a pre-processing part with a minimal duration of 1 time unit (between transition x_5 and x_6). Transition x_6 then represents the finish of the pre-processing, but also serves as a transfer transition, which is synchronized with x_2 and x_3, as well as the start of the actual processing. The processing has a minimal time duration of 22 time units, and after finishing the processing the part is transferred back to resource R_1 (transition x_7). When the transfer is complete the post-processing part of activity act_2 starts (transition x_8) and the activity concludes with transition x_9.

The minimal timing information that is enclosed in the timed event graph can be written as a matrix A_s in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. An element $[A_s]_{ij}$ of this matrix represents the dependency of transition x_i from transition x_j. For our example the non-ε elements of matrix A_s are:

$[A_s]_{2,1} = \delta^7,\ [A_s]_{3,2} = e,\quad [A_s]_{3,6} = e,\quad [A_s]_{4,3} = \delta^1,$
$[A_s]_{6,2} = e,\quad [A_s]_{6,5} = \delta^1,\ [A_s]_{7,6} = \delta^{22}\ \ [A_s]_{8,7} = e,$
$[A_s]_{8,11} = e,\quad [A_s]_{9,8} = \delta^1,\ [A_s]_{11,7} = e,\quad [A_s]_{11,10} = \delta^1,$

and $[A_s]_{12,11} = \delta^7$, where the element $[A_s]_{2,1} = \delta^7 = \gamma^0\delta^7$ means that the earliest possible time for the k-th firing of transition x_2 is 7 time units after transition x_1 has fired for the k-th time. The term γ^0 therefore means that both transitions belong to the manufacturing process of the same part. Consequently, the firing instants of transitions $x \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ of one part have to satisfy the inequality

$$x \succeq A_s \otimes x. \tag{2}$$

B. Model of the Cyclic Process Behavior

The timed event graph in Fig. 2 shows the manufacturing process for one part. However, in general production systems are designed to produce a large number of identical parts. Therefore, the process of manufacturing one part is repeated. Note that we only consider manufacturing systems that produce identical parts in a repetitive fashion, i.e., the required operation of the system is cyclic.

For the cyclic operation of the system the user has to define additional specifications. For our running example these specifications are dependencies between the start and finish transitions of activities which are executed on the same resource. For safety reasons the user may (for example) specify that at least two time units have to elapse in between two activities executed on the same resource. As every resource has a capacity of one, this implies that no activity starts earlier than two time units after the same activity of the previous part is finished, i.e.,

$$\begin{aligned} x_1 &\succeq \gamma^1\delta^2 \otimes x_4 \\ x_5 &\succeq \gamma^1\delta^2 \otimes x_9 \\ x_{10} &\succeq \gamma^1\delta^2 \otimes x_{12}. \end{aligned}$$

Moreover, since activities act_1 and act_3 are executed on the same resource, we have to require that act_1 may not begin before act_3 for the previous part has finished and 2 time units have elapsed. Likewise, act_3 may not start before act_1 for the same part has ended and 2 time units have passed:

$$\begin{aligned} x_1 &\succeq \gamma^1\delta^2 \otimes x_{12} \\ x_{10} &\succeq \delta^2 \otimes x_4. \end{aligned}$$

In general, all specifications defined for the repetitive behavior of the system can be summarized in a matrix $A_c \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ such that

$$x \succeq A_c \otimes x. \tag{3}$$

Combining the specifications of the manufacturing process for a single part (2) with the specifications of the cyclic behavior of the manufacturing process (3), one obtains the inequality

$$x \succeq \underbrace{(A_s \oplus A_c)}_{A} \otimes x. \tag{4}$$

Note that the smallest x satisfying this inequality contains the earliest possible firing instants of the corresponding transitions. Accordingly, all activities finish as early as possible with respect to matrices A_s and A_c.

IV. Control of Cyclic Manufacturing Systems

With the information given in matrix A (4) and some additional information on control inputs and outputs, it is straightforward to implement a feedforward control. However, thanks to residuation theory it is also possible to determine a feedback controller, which can then act on possible deviations from the optimal operation that occur during run-time (see e.g., [6] and [8]).

For manufacturing systems one can usually control the start events of every single activity. Thus, in our running example there would be three inputs u_1, u_2, and u_3, which act directly on the start events x_1, x_5, and x_{10}, i.e., $x_1 \succeq u_1$, $x_5 \succeq u_2$, and $x_{10} \succeq u_3$. Usually, manufacturing systems are operated at the highest possible throughput to produce as many parts in as little time as possible. Consequently, control in this framework is restricted to delaying the occurrence of these events, i.e., the firing of the corresponding transitions. The output y may be, for example, the finish transition of the last activity of a single part, i.e., $y = x_{12}$. Then the system equations are

$$x \succeq Ax \oplus Bu \tag{5}$$

$$y = Cx \tag{6}$$

with $x \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{12\times 1}$, $u \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{3\times 1}$, $y \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{1\times 1}$, $A \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{12\times 12}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{12\times 3}$, and $C \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{1\times 12}$. Using the star operator in dioids, the least solution of (6) can be written as $x = A^*Bu$. Therefore, $y = CA^*Bu = Gu$, where matrix G represents the input/output relation of the system. The elements of G are periodic series in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

A possible state feedback is $u = Fx \oplus v$, where F is the controller and v is an external input, which represents, for example, the arrival of raw materials needed to manufacture a part. Applying this feedback, the closed loop equations are

$$\begin{aligned} x &= (A \oplus BF)\, x \oplus Bv \\ &= (A \oplus BF)^*\, Bv \end{aligned} \tag{7}$$

$$y = Cx = C\,(A \oplus BF)^*\, Bv \tag{8}$$

Using standard properties of the star operator in dioids [5], (8) is equivalent to

$$y = \underbrace{CA^*B\,(FA^*B)^*}_{H}\, v. \tag{9}$$

Thus, $H \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{1\times 3}$ can be seen as the transfer matrix describing the input/output relation of the controlled system.

The purpose of feedback control is to avoid unnecessary accumulation of raw material in the production process, i.e., sub-processes should start just-in-time. Thus, the aim of control is to delay the start of activities as long as possible, while maintaining the maximal throughput of the system. Formally, one has to find the greatest F such that

$$CA^*B\,(FA^*B)^*\, v \preceq CA^*Bv \quad \forall v$$

which is equivalent to

$$CA^*B\,(FA^*B)^* \preceq CA^*B. \tag{10}$$

Applying residuation theory (see, e.g., [6] for details) provides

$$F \preceq (CA^*B) \,\char`\\\!\!\!\circ\, (CA^*B) \,\phi\, (A^*B)\,, \tag{11}$$

and the desired greatest feedback controller is

$$F_{opt} = (CA^*B) \circ\!\!\!\!\diagdown (CA^*B) \phi (A^*B) \,. \tag{12}$$

This controller can be determined using existing software for manipulating periodic series (e.g., [9]). The resulting transfer matrix H for our example is

$$\begin{aligned} H &= CA^*B\,(FA^*B)^* \\ &= \left[\ \delta^{36}\left(\gamma^1\delta^{38}\right)^* \quad \delta^{30}\left(\gamma^1\delta^{38}\right)^* \quad \delta^{8}\left(\gamma^1\delta^{38}\right)^*\ \right], \end{aligned} \tag{13}$$

which indicates that the throughput of the system is one piece every 38 time units.

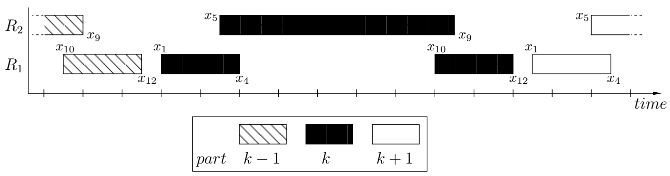

Fig. 3. Gantt chart of the repetitive behavior of the manufacturing process running with maximal throughput and in the just-in-time scheme.

V. Extension to nested Manufacturing Processes

It could be possible that due to a rising demand the manufacturing company would like to increase the production rate of parts produced in our example. Several options are possible to achieve this goal. The easiest way would be to add a second processing unit of two resources which works independently of the first one. This would double the overall production rate. However, looking at the Gantt chart displaying the maximal throughput just-in-time manufacturing process (Fig. 3), one may recognize that the bottleneck in the process is the long execution time on resource R_2. Thus, the company may consider to increase the capacity of this resource, by either exchanging resource R_2 with capacity one with a similar resource of capacity two or by adding a resource (identical) to R_2. The latter option basically generates a resource R_2 with capacity two. Note that, a resource of capacity two can handle two pieces at the same time.

Optimizing the schedule for the extended production system by existing optimization algorithms, e.g., [10], determines a maximal throughput or production rate of two pieces every 40 time units. Thus, the production rate can be almost doubled, but only one additional resource has to be bought. The optimization is performed offline and determines an optimal sequence of activities for every resource. For resource R_2 this sequence of activities is

$$R_2 :\ \ldots, act_2(k-1), act_2(k), act_2(k+1), \ldots$$

The optimal sequence of activities executed on Resource R_1 is

$$R_1 :\ \ldots, act_3(k-2), act_1(k), act_3(k-1), act_1(k+2), \ldots$$

It is important to realize that the production processes of parts may be nested, i.e., some activities of part (k) take place prior to some activities of the previous part $(k-1)$ or, in general, of part $(k-\alpha)$ with $0 < \alpha < \infty$. Such cyclic schedules are called overtaking or interleaving [2]. This means that, normally, more than one part is present in the system at the same time.

To operate this extended manufacturing system, a new controller has to be determined, based on new specifications. Note that the minimal timing information and the sequence of activities to process a single part does not change, i.e., the matrix A_s does not change. The specifications for the cyclic behavior, however, do change due to the modified sequence of activities executed on the two resources.

First of all, the dependencies between the start and finish events of every activity have to be adapted according to the capacities of the resources. While the dependencies for activity act_1 and act_3 remain unchanged, they have to be changed for activity act_2 to match the "new" capacity of resource R_2, i.e.,

$$\begin{aligned} x_1 &\succeq \gamma^1\delta^2 \otimes x_4 \\ x_5 &\succeq \gamma^2\delta^2 \otimes x_9 \\ x_{10} &\succeq \gamma^1\delta^2 \otimes x_{12}. \end{aligned}$$

Also the dependencies between the start and finish events of the different activities executed on resource R_1 have to be adapted according to their new sequence of execution.

With the optimal sequence of activities one can perceive that event x_1 of the k-th part is preceded by event x_{12} of the $(k-2)$-nd part. Formally, this can be written as $x_1 \succeq \gamma^2\delta^2 \otimes x_{12}$, where γ^2 indicates the difference in the part number and δ^2 represents the two time units (as specified by the user), which have to elapse between the finish and start events of subsequently executed activities.

For the dependencies between the end of activity act_1 and the start of act_3 the nested character of the schedule is visible, i.e., activity act_3 of part k starts after activity act_1 of the $(k+1)$-st part is finished. Formally this can be written as $x_{10} \succeq \gamma^{-1}\delta^2 x_4$. Assume that the user needs to impose additional constraints, e.g., by giving upper bounds for the duration of activities or by providing maximal times between the occurrence of events in different activities. As a specific example assume that activity act_1 of the $(k+1)$-st part needs to be finished at least two time units before the processing of activity act_2 of the k-th part is finished, i.e., transition x_7 is fired. These dependencies can be rewritten as

$$\begin{aligned} \delta^2 \otimes x_4 &\preceq \gamma^1 x_{10} \\ \delta^2 \otimes x_4 &\preceq \gamma^1 x_7 \end{aligned}$$

Then, by introducing a new additional transition (x_{13}) it is possible to split the dependencies to

$$\begin{aligned} x_{13} &\succeq \delta^2 \otimes x_4 \\ x_{13} &\preceq \gamma^1 \otimes x_7 \\ x_{13} &\preceq \gamma^1 \otimes x_{10}. \end{aligned}$$

The latter two inequalities can be merged to

$$x_{13} \preceq \gamma^1 \otimes x_7 \wedge \gamma^1 \otimes x_{10},$$

where $\wedge$ is the greatest lower bound, and which is equivalent to (see Sec. II-A)

$$x_{13} \preceq \gamma^1 \odot x_7 \wedge \gamma^1 \odot x_{10}.$$

The specifications of the desired operation can then be summarized in matrix form, i.e.,

$$x \succeq \underline{A}_c \otimes x$$
$$x \preceq \overline{A}_c \odot x,$$

where x is the extended system vector, matrix $\underline{A}_c$ represents the standard dependencies, and matrix $\overline{A}_c$ represents the additional dependencies. Note that elements $[\underline{A}_c]_{ij} = \varepsilon = \gamma^{+\infty}\delta^{-\infty}$, unless there is a dependency between x_i and x_j. For matrix $\overline{A}_c$ all elements are $\top = \gamma^{-\infty}\delta^{+\infty}$ (which is the zero-element of the dual multiplication), except for $[\overline{A}_c]_{13,7} = [\overline{A}_c]_{13,10} = \gamma^1$.

Including the information of the process for a single part we obtain the following system description

$$\underbrace{(A_s \oplus \underline{A}_c)}_{\underline{A}} \otimes x \preceq x \preceq \underbrace{\overline{A}_c}_{\overline{A}} \odot x. \tag{14}$$

Note that, matrix A_s is extended according to the state vector extension, i.e., $A_s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{13\times 13}$ with only zero-elements in the 13-th row and column.

(14) is equivalent to requiring that x is simultaneously in the image of $\underline{A}^*$ and $\overline{A}_*$ [11]. Using the dual residuation, this can be expressed as [11]

$$x = \left(\overline{A}_* \bullet\!\!\!\backslash\, \underline{A}^*\right)^* \otimes x \tag{15}$$
$$= \underline{\overline{A}} \otimes x. \tag{16}$$

The latter equation is similar to (4) and all elements of matrix $\underline{\overline{A}}$ are periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Therefore, a controller can be determined using the same approach as mentioned for the standard system,i.e.,

$$F_n = \left(C\underline{\overline{A}}^*B\right) \circ\!\!\!\!\backslash \left(C\underline{\overline{A}}^*B\right) \circ\!\!\!\!/ \left(\underline{\overline{A}}^*B\right). \tag{17}$$

This controller, then, assures to maintain the throughput while starting all activities just-in-time. The corresponding Gantt chart for the production sequence with resource R_2 being of capacity two is shown in Fig. 4.

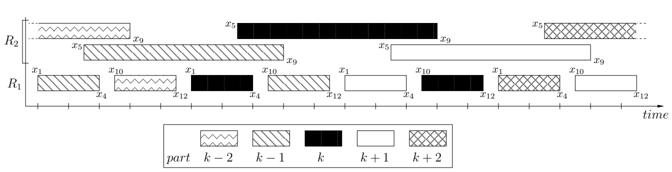

Fig. 4. Gantt chart of the repetitive (just-in-time) behavior of the manufacturing process of identical parts with resource R_2 being of capacity two.

Also the elements of this new controller are periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and one advantage of this is the possibility to easily transform periodic series into a representation, that can be implemented on a standard PLC (see [1] for details).

VI. Conclusion

This paper addresses the modeling of cyclically operated manufacturing systems in a dioid setting. For these manufacturing systems we consider processes, in which parts may visit the same resource more than once. It has been shown how the obtained model can be used to determine a feedback controller, that maintains the throughput while the system operates in the just-in-time scheme. Often manufacturing systems are subject to modifications due to changes in the customer demand. For such a case, we show, how an existing model can be extended according to the extension of the original system. Using the dual multiplication of the dioid and the corresponding dual residuation, a model for systems operated with a nested schedule can be written in the same framework. Therefore, the proposed approach is flexible and can be applied to a large number of different systems.

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