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Consensus-Based Synchronizing Control for Networks of Timed Event Graphs

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Abstract— **Consensus algorithms are an attractive approach to perform distributed decision making. To enforce coordination between entities in a network, they need to share information over this network in order to reach a consensus. We propose an approach to apply a consensus-based control protocol to a network of max-plus linear non-autonomous dynamical systems. In this framework the networked control system is required to achieve consensus on the asymptotic growth rate of the outputs. This leads to a synchronized and stable networked system. The conditions needed to achieve consensus and the consensus value are investigated.**

I. Introduction and Motivation

The stabilization problem for a timed event graph (TEG) has been analysed by, e.g., [1], [2], [3] and [4]. Stability is defined as the property that the number of tokens in the TEG is bounded. In [5] it was shown that for a class of TEGs, stability can be achieved using output feedback.

In order to investigate a network of TEGs, we use a framework based on cooperation between network members. Cooperation in a network can occur if its elements share information over the network, and develop a consistent view regarding objectives and relevant information on the environment. A wide range of different applications of distributed cooperative systems have been reported in the literature (e.g., [6], [7] and [8]). In some applications, a special class of consensus algorithms called max-consensus is used. In this algorithm, the update rule is to take the maximum of all communicated information. An approach to analyze max-consensus which is based on max-plus algebra (e.g., [5] and [9]) was introduced in [10] and [11]. In this framework, max-consensus algorithms become (piecewise) linear.

The aim of our contribution is to use cooperation between network members, each modelled as a TEG, in order to synchronize the firing rate of the output transitions. We consider a class of structurally controllable and observable TEGs and investigate the case of fixed communication topologies. Depending on the communication topology, the earliest firing time of the input transitions of a TEG can be affected by the earliest firing time of the output transitions of other TEGs. We investigate under which conditions a consensus is achieved and show that the networked system can be stabilized using a consensus-based control.

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The paper is organized as follows: Section II summarizes the necessary elements from graph theory, Petri net theory and max-plus algebra. Section III recapitulates state space representations for TEGs in a max-plus framework. It discusses controllability and observability issues and reduced state space representations. Section IV introduces the problem setup of consensus-based control for a group of TEGs in a max-plus algebraic setting. Section V provides convergence conditions for achieving a consensus and investigates the convergence value. Finally some simulation results and a conclusion are presented in Section VI and VII.

II. Graph Theory, Max-Plus Algebra and Timed Event Graphs

A. Graph Theory

Information exchange between nodes in a network is modeled by means of *directed* or *undirected graphs*. A directed graph $\mathcal{G}$ is a pair $(\mathcal{N},\mathcal{E})$, where $\mathcal{N}=\{1,\cdots,n\}$ is a finite nonempty node set and $\mathcal{E}\subseteq\mathcal{N}\times\mathcal{N}$ is a set of ordered pairs of nodes, called edges. Additionally, in weighted graphs each edge is equipped with a weight. Existence of an edge $(j,i)\in\mathcal{E}$ denotes that node i can obtain information from node j. If (j,i) is an edge in a directed graph, node j is called a *predecessor* of node i. $\mathcal{J}_i$ is the set of predecessor nodes of node i, i.e., $\mathcal{J}_i=\{j\in\mathcal{N}|(j,i)\in\mathcal{E}\}$. A *path* is a sequence of nodes $(i_1,\cdots,i_p)$, $p>1$, such that i_j is a predecessor of i_{j+1}, $j=1,\cdots,p-1$. An *elementary path* is a path in which no node appears more than once. If the initial and final node of a path coincide, it is called a *circuit*. If the path $(i_1,\cdots,i_{p-1})$ of the circuit $(i_1,\cdots,i_{p-1},i_p=i_1)$ is elementary, one speaks of an *elementary circuit*. A directed graph is said to be *strongly connected* if there is a path from any node to any other node in the graph, and it is called *connected* if the graph obtained by adding an edge (i,j) for every existing edge (j,i) in the original graph is strongly connected. A sub-graph of a graph $(\mathcal{N},\mathcal{E})$ is a graph $(\mathcal{N}',\mathcal{E}')$, with $\mathcal{N}'\subset\mathcal{N}$, $\mathcal{E}'\subseteq\mathcal{N}'\times\mathcal{N}'$ and $\mathcal{E}'\subseteq\mathcal{E}$. A strongly connected sub-graph is denoted by *SCS*. The *union graph* $\bigcup_i\mathcal{G}_i$ of the directed graphs $\mathcal{G}_i$ $(\mathcal{N}_i,\mathcal{E}_i)$ is the graph $(\mathcal{N},\mathcal{E})$, where

$$\begin{aligned}\mathcal{N} &:= \bigcup_i\mathcal{N}_i\,,\\ \mathcal{E} &:= \bigcup_i\mathcal{E}_i\,.\end{aligned}$$

More details on graph theory can be found in, e.g., [12], [5], [2] and [9].

B. Petri Nets and Timed Event Graphs

In the following, we summarize some basic definitions and facts on Petri nets and timed event graphs (see, e.g., [5], [2] and [13]).

A Petri net graph is a directed bipartite graph $N = (P, T, E, w)$, where $P = \{p_1, \cdots, p_n\}$ is the (finite) set of places, $T = \{t_1, \cdots, t_m\}$ is the (finite) set of transitions, $E \subseteq (P \times T) \cup (T \times P)$ is the set of directed edges from places to transitions and from transitions to places, and $w : (P \times T) \cup (T \times P) \to \mathbb{N}_0$ is a weight function, with $w(p_i, t_j) > 0$, iff $(p_i, t_j) \in E$ and $w(t_j, p_i) > 0$, iff $(t_j, p_i) \in E$. A Petri net (N, M^0) is a pair consisting of a Petri net graph and a vector of initial markings $M^0 \in \mathbb{N}_0^n$, $n = |P|$. p_i is an upstream place of transition t_j (and t_j is a downstream transition of place p_i), if $(p_i, t_j) \in E$. Conversely, p_i is a downstream place of transition t_j (and t_j is an upstream transition of place p_i), if $(t_j, p_i) \in E$. A transition t_j can fire, if the number of tokens M_i, in each upstream place is at least $w(p_i, t_j)$. If a transition t_j fires, the number of tokens in place p_i changes according to

$$M_i' = M_i + w(t_j, p_i) - w(p_i, t_j)\,, \tag{1}$$

where M_i and M_i' refer to the number of tokens in place p_i before and after the firing of transition t_j.

A timed Petri net with holding times is a triple (N, M^0, π), where (N, M^0) is a Petri net and $\pi \in \mathbb{R}^n$ represents the holding times of the places, i.e., π_i is the time a token has to remain in place p_i before it contributes to enabling downstream transitions of p_i.

A(n) (timed) event graph is a (timed) Petri net with exactly one upstream and one downstream transition for each place and all weights equal to 1. The set of transitions of a timed event graph (TEG) can be partitioned into input, output and internal transitions. An input transition is a transition with no upstream place, an output transition is a transition with no downstream place. All other transitions are internal transitions.

Stability of a TEG means that the number of tokens in any place is bounded. A TEG is called *structurally controllable*, if there is a path to every internal transition of the graph originating in the set of input transitions. A TEG is said to be *structurally observable*, if, from every internal transition there exists a path to at least one output transition ([2]).

C. Max-Plus Algebra

Max-plus algebra, e.g., [5], [9], represents a powerful tool for simulation and analysis of a class of timed discrete-event systems and allows for a compact representation of weighted graphs.

Max-plus algebra consists of two binary operations, $\oplus$ and $\otimes$, on the set $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty, +\infty\}$. The operations are defined as follows:

$$a \oplus b \;:=\; \max\{a, b\}, \tag{2}$$

$$a \otimes b \;:=\; a + b. \tag{3}$$

The neutral element of max-plus addition $\oplus$ is $-\infty$, denoted as ε. The neutral element of multiplication $\otimes$ is 0, denoted as e. The elements ε and e are also referred to as the zero and one element of max-plus algebra. Similar to conventional algebra, associativity and commutativity of addition and multiplication, and distributivity of multiplication over addition also hold for the max-plus algebra. In contrast to standard algebra, addition is idempotent, i.e., $a \oplus a = a$, $\forall a \in \mathbb{R}_{\max}$. Both operations can be extended to matrices in a straightforward way. For $A, B \in \mathbb{R}_{\max}^{m \times n}$,

$$(A \oplus B)_{ij} := a_{ij} \oplus b_{ij}, \quad i = 1, \cdots, m,\ j = 1, \cdots, n\,.$$

For $A \in \mathbb{R}_{\max}^{m \times n},\ B \in \mathbb{R}_{\max}^{n \times q}$,

$$(A \otimes B)_{ij} := \bigoplus_{k=1}^{n} (a_{ik} \otimes b_{kj}) = \max_k (a_{ik} + b_{kj}),$$
$$i = 1, \cdots, m,\ j = 1, \cdots, q.$$

Multiplication of a matrix $A \in \mathbb{R}_{\max}^{m \times n}$ and a scalar $\alpha \in \mathbb{R}_{\max}$ is defined by

$$(\alpha \otimes A)_{ij} := \alpha \otimes a_{ij} = \alpha + a_{ij},$$
$$i = 1, \cdots, m,\ j = 1, \cdots, n.$$

Note that, as in conventional algebra, the multiplication symbol $\otimes$ is often omitted. In the sequel, we denote a matrix of zero elements of appropriate dimension by N. The identity matrix I is a square matrix with

$$(I)_{ij} := \begin{cases} \mathrm{e} & \text{for } i = j\,, \\ \varepsilon & \text{else}\,. \end{cases}$$

For any matrix $A \in \mathbb{R}_{\max}^{n \times n}$, its *precedence graph* $\mathcal{G}(A)$ is defined in the following way: it has n nodes, denoted by $1, \cdots, n$, and (j, i) is an edge if and only if $a_{ij} \neq \varepsilon$. In this case a_{ij} is the weight of edge (j, i). Then

- A path in $\mathcal{G}(A)$ is a sequence of $p > 1$ nodes, denoted by $\rho := i_1, \ldots, i_p$, such that $a_{i_{k+1} i_k} \neq \varepsilon$, $k = 1, \ldots, p-1$.
- The weight of a path $\rho := i_1, \ldots, i_p,$, denoted by $|\rho|_w$, is defined as $|\rho|_w := \sum_{k=1}^{p-1} a_{i_{k+1} i_k}$ and its corresponding length is $|\rho|_l := p - 1$.
- $(A^k)_{ij}$, $k \geq 1$, represents the maximal weight of all paths of length k from node j to node i, and $A^0 = I$.

The matrix A is called *irreducible* if it cannot be brought to upper-block triangular form by identical row and column permutations. A is irreducible if and only if $\mathcal{G}(A)$ is strongly connected.

Given $A \in \mathbb{R}_{\max}^{n \times n}$, scalars λ and vectors ν satisfying

$$A \otimes \nu = \lambda \otimes \nu,$$

are referred to as eigenvalues and eigenvectors of A. If A is irreducible, it possesses a unique eigenvalue which can be computed as ([14])

$$\lambda = \bigoplus_{i=1}^{n} (tr(A)^i)^{1/i}\,. \tag{4}$$

$tr(A)$ denotes the trace of A, i.e.,

$$tr(A) = \bigoplus_{i=1}^{n} a_{ii}\,.$$

$\alpha^{1/i}$ is the i-th root of α, i.e,

$$(\alpha^{1/i})^i = \underbrace{\alpha^{1/i} \otimes \cdots \otimes \alpha^{1/i}}_{i\text{-times multiplication}} = \alpha\,.$$

(4) implies that the the eigenvalue of an irreducible matrix A represents the maximal mean weight of all circuits in $\mathcal{G}(A)$.

Let $\{x(k) : k \in \mathbb{N}\}$ be a sequence generated by

$$x(k+1) = A \otimes x(k)\,, \quad k \geq 0\,,$$

where $A \in \mathbb{R}_{\max}^{n\times n}$ and $x_0 = (x_1(0), \cdots, x_n(0))^T$ is the vector of initial condition. Hence,

$$x(k) = A^k \otimes x_0\,, \quad \forall k \geq 0\,.$$

To convey the dependency of $x(k)$ on its initial value we write $x(k; x_0)$.

Definition 1: ([9]) Given a sequence $\{x(k) : k \in \mathbb{N}\}$ in $\mathbb{R}_{\max}^n$ and assuming that for all $i \in \{1, \cdots, n\}$

$$\eta_i = \lim_{k\to\infty} \frac{x_i(k)}{k}$$

exists[1], the vector $\eta = (\eta_1, \cdots, \eta_n)^T$ is called the *cycle-time* vector of the sequence $\{x(k) : k \in \mathbb{N}\}$. $\triangleleft$

If all the η_i's have the same value, this value is called the *asymptotic growth rate* of the sequence $\{x(k) : k \in \mathbb{N}\}$ and is denoted by η.

III. Modelling and System Properties

Consider an arbitrary TEG (N, M^0, π). Introduce $\tau_j(k)$ as the earliest possible time for the k-th firing of transition t_j and $\mu_i(k)$ as the earliest possible time for place p_i to receive its k-th token. Denote the set of upstream places of a transition t_j by $I(t_j)$ and the set of upstream transitions of a place p_i by $I(p_i)$. Recall that in a TEG $|I(p_i)| = 1$. If $|I(t_j)| > 0$, i.e., if t_j is not an input transition, we can write

$$\begin{aligned} \mu_i(k + M_i^0) &= \tau_r(k)\,, \quad t_r \in I(p_i)\,,\ k = 1, 2, \cdots\,, \\ \tau_j(k) &= \max_{i, p_i \in I(t_j)} \{\mu_i(k) + \pi_i\}\,,\ k = 1, 2, \cdots\,, \end{aligned}$$

and, in the max-plus algebra,

$$\begin{aligned} \mu_i(k + M_i^0) &= \tau_r(k)\,, \quad t_r \in I(p_i)\,,\ k = 1, \cdots\,, & (5) \\ \tau_j(k) &= \bigoplus_{i, p_i \in I(t_j)} (\pi_i \otimes \mu_i(k))\,,\ k = 1, \cdots\,. & (6) \end{aligned}$$

Eliminating μ_i, $i = 1, \cdots, n$, from (5) and (6) provides an explicit relation for the firing instants of transitions. If t_j is an input transition, then its firing times are determined by the environment.

We partition the set of transitions in the TEG (N, M^0, π) as $T = T_I \cup T_O \cup T_{int}$, where T_I, T_O and T_{int} are the set of input, output and internal transitions, respectively. Furthermore, $|T_I| = m$, $|T_O| = q$ and $|T_{int}| = n'$.

We denote the vectors of earliest firing times of internal transitions, input transitions and output transitions by $\bar{x}$, u and $\bar{y}$, respectively. We can then write the equations for the earliest firing times of transitions as follows

$$\begin{aligned} \bar{x}(k) &= A_0\bar{x}(k) \oplus A_1\bar{x}(k-1) \oplus \cdots \oplus A_\sigma \bar{x}(k-\sigma) \\ &\quad \oplus B_0 u(k) \oplus \cdots \oplus B_\sigma u(k-\sigma)\,, & (7) \\ \bar{y}(k) &= C_0\bar{x}(k) \oplus \cdots \oplus C_\sigma \bar{x}(k-\sigma) \\ &\quad \oplus D_0 u(k) \oplus \cdots \oplus D_\sigma u(k-\sigma)\,, & (8) \end{aligned}$$

where $A_i \in \mathbb{R}_{\max}^{n'\times n'}$, $B_i \in \mathbb{R}_{\max}^{n'\times m}$, $C_i \in \mathbb{R}_{\max}^{q\times n'}$, and σ denotes the maximal initial marking, i.e., $\sigma = \max_i\{M_i^0\}$. The least solution of (7) is provided by

$$\begin{aligned} \bar{x}(k) &= \tilde{A}_1\bar{x}(k-1) \oplus \cdots \oplus \tilde{A}_\sigma \bar{x}(k-\sigma) \\ &\quad \oplus \tilde{B}_0 u(k) \oplus \cdots \oplus \tilde{B}_\sigma u(k-\sigma)\,, & (9) \end{aligned}$$

where $\tilde{A}_i = A_0^* A_i$, $i = 1, \cdots, \sigma$ and $\tilde{B}_j = A_0^* B_j$, $j = 0, \cdots, \sigma$, and $A_0^* = \bigoplus_{i\geq 0} A_0^i$. If $\mathcal{G}(A_0)$ does not contain any circuits with positive weight, $A_0^* = \bigoplus_{i=0}^{n'-1} A_0^i$. Defining

$$\tilde{x}(k) = (\bar{x}^T(k) \cdots \bar{x}^T(k-\sigma+1)\ u^T(k) \cdots u^T(k-\sigma+1))^T\,,$$

we get the following state space representation for the TEG (N, M^0, π)

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}\tilde{x}(k) \oplus \tilde{B}u(k+1)\,, & (10) \\ \tilde{y}(k) &= \bar{y}(k) = \tilde{C}\tilde{x}(k)\,, & (11) \end{aligned}$$

with

$$\tilde{A} = \begin{pmatrix} \tilde{A}_1 & \cdots & \tilde{A}_\sigma & \tilde{B}_1 & \cdots & \tilde{B}_\sigma \\ & & N & & & N \\ & I & \vdots & & N & \vdots \\ & & N & & & N \\ N & \cdots & N & & \cdots & N \\ & & N & & & N \\ & N & \vdots & & I & \vdots \\ & & N & & & N \end{pmatrix}, \tilde{B} = \begin{pmatrix} \tilde{B}_0 \\ N \\ \vdots \\ N \\ I \\ N \\ \vdots \\ N \end{pmatrix},$$

and

$$\tilde{C} = \begin{pmatrix} C_0 & \cdots & C_\sigma & D_0 & \cdots & D_\sigma \end{pmatrix},$$

where I and N are identity and zero matrices of appropriate dimensions, respectively, and $\tilde{A} \in \mathbb{R}_{\max}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbb{R}_{\max}^{\tilde{n}\times m}$ and $\tilde{C} \in \mathbb{R}_{\max}^{q\times\tilde{n}}$, with $\tilde{n} = (n' + m)\sigma$.

The max-plus linear system (10) and (11) corresponds to a weighted graph $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ with $\tilde{n}$ internal nodes, m input nodes and q output nodes. If $\tilde{a}_{ij} \neq \varepsilon$, there is an edge with the weight $\tilde{a}_{ij}$ from the internal node j to the internal node i. If $\tilde{b}_{il} \neq \varepsilon$, there is an edge with the weight $\tilde{b}_{il}$ from the input node l to the internal node i and if $\tilde{c}_{kj} \neq \varepsilon$, there is an edge with the weight $\tilde{c}_{kj}$ from the internal node j to the output node k.

Structural controllability and structural observability of $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ can be defined in complete analogy to TEGs: $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ is said to be structurally controllable, if there exists a path to every internal node starting in an input node; it is said to be structurally observable, if there exists a path from any internal node to an output node.

[1]Note that $\frac{\cdot}{k}$ denotes scalar division in the conventional algebra.

Lemma 1: $i)$ $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ is structurally controllable if and only if the *controllability matrix*

$$\mathcal{C} = \left[\begin{array}{ccccc} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} & \cdots & \tilde{A}^{\tilde{n}-1}\tilde{B} \end{array} \right], \qquad (12)$$

contains at least one non-ε element in each *row*.

$ii)$ $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ is structurally observable if and only if the *observability matrix*

$$\mathcal{O} = \left[\begin{array}{c} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \vdots \\ \tilde{C}\tilde{A}^{\tilde{n}-1} \end{array} \right], \qquad (13)$$

contains at least one non-ε element in each *column*. $\triangleleft$

Proof: Recall that $(\tilde{A}^k)_{ij} \neq \varepsilon$ represents the existence of a path of length k from the internal node j to the internal node i and $\tilde{B}_{il} \neq \varepsilon$ the existence of an edge from the input node l to the internal node i, in $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$. Furthermore, as $\tilde{A} \in \mathbb{R}_{\max}^{\tilde{n}\times\tilde{n}}$, the maximal length of an elementary path in $\mathcal{G}(\tilde{A})$ is equal to $\tilde{n}-1$, and hence, the maximal length of an elementary path from an input node to an internal node in $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ is $\tilde{n}$. Thus, the entries of $\tilde{B}$, $\tilde{A}\tilde{B}$, ..., $\tilde{A}^{\tilde{n}-1}\tilde{B}$, represent the existence of paths of length 1, 2, ..., $\tilde{n}$, from an input node to the internal nodes in $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$. Hence, if the i-th row of $\mathcal{C}$ consists of only ε-entries, there is no path of any length from an input node to the internal node i, hence $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ is not structurally controllable. Conversely, if $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ is not structurally controllable, there is at least one internal node i to which there is no path from an input node, i.e., the i-th row of $\tilde{A}^k\tilde{B}$, $k \geq 0$, contains only ε-entries, i.e., $\mathcal{C}$ has an ε-row.

Recalling the graph interpretation of $\tilde{C}$ in $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$, $ii)$ can be proven similarly. ■

Recall that the maximal length of a path in $\mathcal{G}(\tilde{A})$ is equal to $\tilde{n}-1$. Therefore, if there exists a column containing only ε-entries in $\tilde{C}$, $\tilde{C}\tilde{A} \cdots, \tilde{C}\tilde{A}^{\tilde{n}-1}$, the corresponding entry will be equal to ε in $\tilde{C}\tilde{A}^k$, $\forall k$. The same holds for an ε-row in $\tilde{A}^k\tilde{B}$, $\forall k$.

Corollary 1: If a TEG is structurally controllable, the graph $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ corresponding to the state space representation (10) and (11) is also structurally controllable. $\triangleleft$

Proof: Denote the set $\{1, \cdots, (n'+m)\sigma\}$ by Φ. Assume that $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ is not structurally controllable, i.e., there is at least one internal node i in $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$, corresponding to $\tilde{x}_i$, to which there is not path from an input node. Hence, the i-th row of $\mathcal{C}$ is equal to N. Based on the structure of $\tilde{A}$ and $\tilde{B}$ it is clear that there is always a path from an input node to the nodes j, $j \in \Upsilon = \{n'\sigma+1, \ldots, (n'+m)\sigma\}$, hence $i \in \Phi \setminus \Upsilon$. Furthermore, $\tilde{A}$ indicates that the node set $\Phi \setminus \{\Upsilon \cup \{1, \cdots, \tilde{n}\}\}$ only have paths from the input nodes via the nodes in the set $\{1, \cdots, n'\}$. Thus, the assumption is that the node i to which there exists no path from an input node is in the set $\{1, \cdots, n'\}$. Consequently, all the nodes in the set $\{i, n'+i, \cdots, n'(\sigma-1)+i\}$, have no path from the input nodes. This directly contradicts the assumption of structural controllability of the TEG, since the $k-r$-th, $r = 0, \cdots, \sigma-1$, times of firing of the transition t_i (corresponding to $\tilde{x}_l$, $l = i, n'+i, \cdots, n'(\sigma-1)+i$) do not depend on the firing time of any of input transitions. ■

As the following example shows, structural observability of a TEG does not imply structural observability of the graph $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ corresponding to the state space representation (10) and (11).

Example 1: Consider the TEG 1 in Figure 1. Obviously, it is a structurally observable TEG. It is easy to check that in the state space representation of the form (10) and (11), i.e.,

$$\begin{aligned} \tilde{x}(k+1) &= \left(\begin{array}{ccccc} \varepsilon & \varepsilon & \varepsilon & 2 & 3 \\ \varepsilon & \varepsilon & \varepsilon & 8 & 9 \\ \mathrm{e} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \mathrm{e} & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array} \right) \tilde{x}(k) \oplus \left(\begin{array}{c} \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \mathrm{e} \end{array} \right) u(k+1), \\ y(k) &= \left(\begin{array}{ccccc} \varepsilon & \mathrm{e} & \varepsilon & \varepsilon & \varepsilon \end{array} \right) \tilde{x}(k). \end{aligned}$$

the first and third column of $\mathcal{O}$ consist of ε-entries only. $\triangleleft$

Remark 1: A similar discussion on the link between controllability and observability matrices and structural controllability and structural observability of TEGs is given in [15]. But contrary to our approach, where arbitrary initial markings are considered, the results in [15] hold only for the case of zero initial markings. See Example 1 as a counter-example to Property 2 in [15]. $\triangleleft$

Consider a structurally controllable and structurally observable TEG with the state space representation (10) and (11). It is not surprising that entries in $\tilde{x}$ that correspond to internal nodes in $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ from which there is no path to an output node can be eliminated without affecting the input/output behaviour and therefore the earliest firing time of the underlying TEG's output transitions. For this, we introduce an $\tilde{n} \times n$-matrix Q, where Q is obtained from the $\tilde{n} \times \tilde{n}$ identity matrix by removing columns i, if the i-th column of $\mathcal{O}$ contains only ε-entries.

Defining a reduced state vector

$$x(k) = Q^T \tilde{x}(k),$$

and matrices

$$\begin{aligned} A &= Q^T \tilde{A} Q, \\ B &= Q^T \tilde{B}, \\ C &= \tilde{C} Q, \end{aligned}$$

we obtain the state space representation

$$\begin{aligned} x(k+1) &= Ax(k) \oplus Bu(k+1), \qquad (14) \\ y(k) &= Cx(k), \qquad (15) \end{aligned}$$

where $A \in \mathbb{R}_{\max}^{n\times n}$, $B \in \mathbb{R}_{\max}^{n\times m}$ and $C \in \mathbb{R}_{\max}^{q\times n}$.

Lemma 2: The graph $\mathcal{G}(A, B, C)$ is structurally controllable and structurally observable. Furthermore, with $x(0) = Q^T\tilde{x}(0)$, the output sequence $\{\tilde{y}(k) : k \geq 0\}$ and $\{y(k) : k \geq 0\}$ coincide for every input sequence $\{u(k) : k \geq 0\}$. $\triangleleft$

Proof: Recalling that $\mathcal{G}(A, B, C)$ is constructed by removing the internal nodes of $\mathcal{G}(\tilde{A}, \tilde{B}, \tilde{C})$ from which there is no path to an output node, structural observability of $\mathcal{G}(A,B,C)$ is straightforward. Structural controllability of $\mathcal{G}(A,B,C)$ follows from structural controllability of $\mathcal{G}(\tilde{A},\tilde{B},\tilde{C})$, where to all internal nodes there exists a path from an input node. Let i be the only internal node from which there is no path to an output node. Furthermore, assume that removing i results in losing the path from the input node to another internal node j. Hence, there exists a path from node i to node j in $\mathcal{G}(\tilde{A},\tilde{B},\tilde{C})$. Since there is a path from node j to an output node, node i has also a path to an output node which contradicts the assumption.

Recursive application of (10) provides

$$\tilde{x}(k) = \tilde{A}^k \tilde{x}(0) \oplus \bigoplus_{l=0}^{k-1} \tilde{A}^l \tilde{B} u(k-l)\,, \quad k > 0\,,$$

with $\tilde{x}(0)$ an initial state. Hence, applying (11) we get

$$\tilde{y}(k) = \tilde{C}\tilde{A}^k \tilde{x}(0) \oplus \bigoplus_{l=0}^{k-1} \tilde{C}\tilde{A}^l \tilde{B} u(k-l)\,, \quad k > 0\,. \qquad (16)$$

Similarly, (14) and (15) lead to

$$y(k) = CA^k x(0) \oplus \bigoplus_{l=0}^{k-1} CA^l B u(k-l)\,, \quad k > 0\,. \qquad (17)$$

We have to show that the right-hand sides of (16) and (17) are equal for the same input signal $u(k)$ and for all $k > 0$. For $k = 0$ we get from (11) $\tilde{y}(0) = \tilde{C}\tilde{x}(0)$. Since in $\mathcal{O}$, and hence in $\tilde{C}$, there is at least one column i of ε-entries, we can remove these columns from $\tilde{C}$ and the corresponding states $\tilde{x}_i$ from $\tilde{x}$ without affecting the output value, i.e., $\tilde{y}(0) = \tilde{C}QQ^T\tilde{x}(0) = Cx(0) = y(0)$. For $k = 1$, we use the fact the i-th columns in $\tilde{C}$ and $\tilde{C}\tilde{A}$ contain only ε-entries. We remove the corresponding columns in $\tilde{C}$, the corresponding rows and columns of $\tilde{A}$ and the corresponding rows of $\tilde{x}$ and $\tilde{B}$, i.e.,

$$\begin{aligned} \tilde{y}(1) &= \tilde{C}\tilde{A}\tilde{x}(0) \oplus \tilde{C}\tilde{B}u(1) \\ &= \tilde{C}QQ^T\tilde{A}QQ^T\tilde{x}(0) \oplus \tilde{C}QQ^T\tilde{B}u(1) \\ &= CAx(0) \oplus CBu(1) = y(1)\,. \end{aligned}$$

Using the same argument for $\tilde{y}(2)$, we get:

$$\begin{aligned} \tilde{y}(2) &= \tilde{C}\tilde{A}^2\tilde{x}(0) \oplus \tilde{C}\tilde{B}u(2) \oplus \tilde{C}\tilde{A}\tilde{B}u(1) \\ &= (\tilde{C}\tilde{A})QQ^T\tilde{A}QQ^T\tilde{x}(0) \\ &\quad \oplus \tilde{C}QQ^T\tilde{B}u(2) \oplus (\tilde{C}\tilde{A})QQ^T\tilde{B}u(1) \\ &= \tilde{C}QQ^T\tilde{A}QQ^T\tilde{A}QQ^T\tilde{x}(0) \\ &\quad \oplus \tilde{C}QQ^T\tilde{B}u(2) \oplus \tilde{C}QQ^T\tilde{A}QQ^T\tilde{B}u(1) \\ &= CA^2x(0) \oplus CBu(2) \oplus CABu(1) = y(2)\,. \end{aligned}$$

Since, based on $\mathcal{O}$, the i-th entry of $\tilde{C}\tilde{A}^k$ is equal to ε for all $k \geq 0$, it is easy to check that $\tilde{y}(k) = y(k)$, $\forall k \geq 0$. ■

IV. Problem Setup

Consider a group of agents, each modelled as a structurally controllable and structurally observable TEG, working in parallel. To simplify exposition, we assume that each TEG has precisely one input and one output transition. The network is required to work in a synchronized way, i.e., with the same asymptotic growth rate for the output sequences of all subsystems. A possible centralized solution to this problem would be an output feedback from all output to all input transitions of all TEGs. A distributed approach to the problem would be for each agent to communicate the earliest firing time of its output transition to a subset of the other agents, in order to achieve an agreement on the asymptotic growth rate of output sequences.

The scenario described above can be formalized as follows:

- The set of group members is denoted by $\mathcal{N} = \{1, \cdots, p\}$.
- The communication topology is modelled by a weighted directed graph $\mathcal{G}_{comm} = (\mathcal{N}, \mathcal{E})$, where $(j,i) \in \mathcal{E}$, $j, i \in \mathcal{N}$, means that node (graph member) i receives information from node (graph member) j. The weights of all edges are in $\mathbb{R}$, which illustrates the communication delay between the agents in the network.
- Each TEG i can be described by a state space representation in max-plus algebra:

$$\begin{aligned} x^{(i)}(k+1) &= A_i x^{(i)}(k) \oplus B_i u^{(i)}(k+1)\,, \quad (18) \\ y^{(i)}(k) &= C_i x^{(i)}(k)\,, \quad (19) \end{aligned}$$

  where $i \in \mathcal{N}$ is the agent's index, $x^{(i)}(k) = (x_1^{(i)}(k), \cdots, x_{n_i}^{(i)}(k))^T \in \mathbb{R}_{\max}^{n_i}$ is the state vector of agent i, $u^{(i)}(k) \in \mathbb{R}_{\max}$ is the input and $y^{(i)}(k) \in \mathbb{R}_{\max}$ is its output. Furthermore, A_i, B_i and C_i are in $\mathbb{R}_{\max}^{n_i \times n_i}$, $\mathbb{R}_{\max}^{n_i \times 1}$ and $\mathbb{R}_{\max}^{1 \times n_i}$, respectively. Recall from the previous section, we can always choose A_i, B_i and C_i, such that the graph $\mathcal{G}(A_i, B_i, C_i)$ is structurally controllable and structurally observable.
- The information communicated by agent i is chosen as the earliest k-th firing time of its output transition, i.e., $y^{(i)}(k)$, and its input is updated via the consensus-based control protocol

$$u^{(i)}(k+1) = \max_{j \in \mathcal{J}_i} \{y^{(j)}(k)\}\,, \quad i = 1, \cdots, p\,,$$

  and hence in max-plus algebra

$$= \bigoplus_{j \in \mathcal{J}_i} (y^{(j)}(k))\,, \quad i = 1, \cdots, p\,, \qquad (20)$$

  where $\mathcal{J}_i$ is the set of predecessor nodes of node i in the graph $\mathcal{G}_{comm}$, i.e., $\mathcal{J}_i = \{j | (j,i) \in \mathcal{E}\}$.

Definition 2: Given a directed graph $\mathcal{G}_{comm}$ with p members, each described by (18) and (19), and control rule (20), a vector $(y^{(1)}(k), \cdots, y^{(p)}(k))^T$ and an arbitrary vector of initial conditions $x(0) = (x^{(1)^T}(0), \cdots, x^{(p)^T}(0))^T$, and hence a sequence $\{y(k; x(0)) : k > 0\}$. We denote $\frac{y(k;x(0))}{k}$ by $\xi(k; x(0)) \in \mathbb{R}_{\max}^n$ and refer to this as the consensus variable. A consensus is then said to be achieved, if

$$\lim_{k\to\infty} \xi_i(k; x(0)) = \lim_{k\to\infty} \xi_j(k; x(0)) := \eta\,, \quad \forall i, j \in \mathcal{N}\,, \quad (21)$$

i.e., the sequence $\{y(k) : k > 0\}$, with $y(k) = (y^{(1)}(k), \cdots, y^{(p)}(k))^T$, exhibits an asymptotic growth rate η. ◁

In the following, we will investigate under which conditions the control strategy (20) guarantees consensus.

V. Consensus and Stability

Let $\mathcal{A} \in \mathbb{R}^{p\times p}_{\max}$ be a matrix such that its precedence graph $\mathcal{G}(\mathcal{A}) = \mathcal{G}_{comm}$. Then, with $u(k) := (u^{(1)}(k), \cdots, u^{(p)}(k))^T$, the feedback (20) can be written as

$$u(k+1) = \mathcal{A} \otimes y(k)\,. \tag{22}$$

Defining

$$\overline{A} = blockdiag(A_i)\,, \quad i = 1, \cdots, p\,,$$
$$\overline{B} = blockdiag(B_i)\,, \quad i = 1, \cdots, p\,,$$
$$\overline{C} = blockdiag(C_i)\,,, \quad i = 1, \cdots, p\,,$$

and with $x(k) = (x^{(1)^T}(k), \cdots, x^{(p)^T}(k))^T \in \mathbb{R}^{\bar{n}}_{\max}$, $\bar{n} = \sum_{i=1}^{p} n_i$, the closed loop system can be written as

$$x(k+1) = \underbrace{(\overline{A} \oplus \overline{B}\mathcal{A}\overline{C})}_{\mathfrak{A}} x(k)\,, \tag{23}$$

$$y(k) = \overline{C} \otimes x(k)\,. \tag{24}$$

To show under which condition our proposed approach provides consensus, we need some essential preliminary results.

Fact 1: ([9]) Consider the difference equation $x(k+1) = A \otimes x(k)$, $k \geq 0$, with $A \in \mathbb{R}^{n\times n}_{\max}$ a square matrix and $x(0) = x_0 \in \mathbb{R}^n$ as initial vector. If for a particular x_0

$$\lim_{k\to\infty} \frac{x(k; x_0)}{k}$$

exists, then this limit also exists and has the same value for any other initial vector $\hat{x}_0$ in $\mathbb{R}^n$. ◁
Note that the Fact 1 requires that x_0 and $\hat{x}_0$ are vectors in $\mathbb{R}^n$.

Fact 2: ([2]) A TEG with Petri net graph N under an earliest firing time controlled execution is stable if N is strongly connected. ◁

Using the above issues, we can now state the main results of our contribution.

Theorem 1: Any group of p structurally controllable and structurally observable TEGs is stabilized by consensus-based control in the form of (22), if the communication topology $\mathcal{G}(\mathcal{A})$ is strongly connected. ◁

Proof: If $\mathcal{A}_{ij} \neq \varepsilon$, the output transition of TEG j is connected to the input transition of TEG i via a place with holding time $\mathcal{A}_{ij}$. Structural controllability and structural observability of each TEG imply the existence of a path from its input transition to its output transition. Hence, if $\mathcal{G}(\mathcal{A})$ is strongly connected, the overall Petri net graph corresponding to the closed-loop system is also strongly connected. Hence, stability follows from Fact 2. ■

Theorem 2: Consider a group of p arbitrary structurally controllable and structurally observable TEGs. Furthermore, let $\mathcal{G}_{comm} = \mathcal{G}(\mathcal{A})$ be a directed graph. Using a consensus-based control of form (22), a consensus in terms of Definition 2 is achieved if and only if $\mathcal{A}$ is irreducible, i.e., $\mathcal{G}(\mathcal{A})$ is strongly connected. ◁

Proof: Recall that we can always describe a structurally observable and structurally controllable TEG by a state space representation of form (18) and (19), such that $\mathcal{G}(A^{(i)}, B^{(i)}, C^{(i)})$ is structurally controllable and structurally observable, $\forall i$. Consider (23) and (24) and let $x_0 \in \mathbb{R}^{\bar{n}}$ be a vector of arbitrary initial conditions. We have to show that $\lim_{k\to\infty} \xi(k; x_0)$ exists and $\lim_{k\to\infty} \xi_i(k; x_0) = \lim_{k\to\infty} \xi_j(k; x_0)$, $\forall i, j$.

Obviously, $\mathcal{G}(\mathfrak{A}) = \mathcal{G}(\overline{A}) \cup \mathcal{G}(\overline{B}\mathcal{A}\overline{C})$ with $\bar{n}$ nodes, $\bar{n} = \sum_{i=1}^{p} n_i$. The graph $\mathcal{G}(\overline{A})$ is the union of p sub-graphs $\mathcal{G}(A^{(i)})$, where $\mathcal{G}(A^{(i)})$ is the sub-graph of $\mathcal{G}(A^{(i)}, B^{(i)}, C^{(i)})$ obtained by removing the input and output nodes and the corresponding edges. Each $\mathcal{A}_{ij} \neq \varepsilon$ add edges from the output node of $\mathcal{G}(A^{(j)}, B^{(j)}, C^{(j)})$ to the input node of $\mathcal{G}(A^{(i)}, B^{(i)}, C^{(i)})$. For $\mathcal{G}(\mathfrak{A})$, this corresponds to adding edges from internal nodes of $\mathcal{G}(A^{(j)})$ to internal nodes of $\mathcal{G}(A^{(i)})$ with weights determined by the entries of the matrix $(\mathcal{A})_{ij} B_i C_j \in \mathbb{R}^{n_i \times n_j}$. Note that $B_i C_j \neq N$, $\forall i, j$. Recalling that in each $\mathcal{G}(A^{(i)}, B^{(i)}, C^{(i)})$ there exists always a path from the input node to the output node, strong connectivity of $\mathcal{G}(\mathcal{A})$ directly leads to a strongly connected $\mathcal{G}(\mathfrak{A})$. Thus, $\mathfrak{A}$ is irreducible and has a finite eigenvalue λ and a finite eigenvector $\nu \in \mathbb{R}^{\bar{n}}$. Picking $\nu \in \mathbb{R}^{\bar{n}}$ as a vector of initial conditions x_0

$$\begin{aligned}\lim_{k\to\infty} \xi(k; x_0 = \nu) &= \lim_{k\to\infty} \frac{y(k;\nu)}{k} = \lim_{k\to\infty} \frac{C \otimes x(k;\nu)}{k} \\ &= \lim_{k\to\infty} \frac{C \otimes \mathfrak{A}^k \otimes \nu}{k} \\ &= \lim_{k\to\infty} \frac{C \otimes \lambda^k \otimes \nu}{k}\,,\end{aligned}$$

with $\lambda \in \mathbb{R}$ the unique eigenvalue of $\mathfrak{A}$. Hence, in max-plus algebra we have

$$\begin{aligned}\lim_{k\to\infty} \xi(k;\nu) &= \lim_{k\to\infty} \frac{C \otimes \begin{pmatrix} \nu_1 \otimes \lambda^k \\ \vdots \\ \nu_{\bar{n}} \otimes \lambda^k \end{pmatrix}}{k} \\ &= \lim_{k\to\infty} \frac{\begin{pmatrix} \bigoplus_{i=1}^{\bar{n}} c_{1i} \otimes \nu_i \otimes \lambda^k \\ \vdots \\ \bigoplus_{i=1}^{\bar{n}} c_{pi} \otimes \nu_i \otimes \lambda^k \end{pmatrix}}{k}\,.\end{aligned}$$

Consequently, in the conventional algebra

$$\lim_{k\to\infty} \xi(k;\nu) = \lim_{k\to\infty} \frac{\begin{pmatrix} \max_{i\in\mathcal{I}}\{c_{1i} + \nu_i + k\cdot\lambda\} \\ \vdots \\ \max_{i\in\mathcal{I}}\{c_{pi} + \nu_i + k\cdot\lambda\} \end{pmatrix}}{k}\,,$$

with $\mathcal{I} = 1, \cdots, \bar{n}$. Since ν and λ are finite, we get

$$\lim_{k\to\infty} \xi(k;\nu) = \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix} \in \mathbb{R}^p\,,$$

According to Fact 1 this value is the same for arbitrary initial conditions. Hence, consensus is achieved for arbitrary x_0.

Necessity can be proven by contradiction. Let's assume that $\mathcal{G}(\mathcal{A})$ is not strongly connected and therefore, $\mathfrak{A}$ is not irreducible. Hence, according to Theorem 3.17 in [9], there exists an initial condition $x(0)$, such that the elements of the cycle time vector of $\mathfrak{A}$ are different for arbitrary TEGs, i.e., a consensus is not achieved. ■

The next Corollary discusses the consensus value achieved under the conditions given in Theorem 2.

Corollary 2: Let $\mathcal{A} \in \mathbb{R}_{\max}^{p \times p}$, with $(\mathcal{A})_{ij} \in \{\varepsilon, \mathrm{e}\}$, $\forall i, j$. If a consensus in terms of Definition 2 is achieved, then the consensus value is given by the unique eigenvalue of $\mathfrak{A}$, i.e.,

$$\lambda = \bigoplus_{\kappa=1}^{\bar{n}} (tr(\overline{A} \oplus \overline{B}\mathcal{A}\overline{C})^{\kappa})^{1/\kappa} . \tag{25}$$

◁

Proof: The proof follows directly from Theorem 2 and (4). ■

VI. Example and Simulation Results

Consider the TEGs in Figure 1 and the graph of the communication topology $\mathcal{G}_{comm} = \mathcal{G}(\mathcal{A})$ in Figure 2, with

$$\mathcal{A} = \begin{pmatrix} \varepsilon & \mathrm{e} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \mathrm{e} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \mathrm{e} \\ \mathrm{e} & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} .$$

All the edges in $\mathcal{G}(\mathcal{A})$ are equal to e, i.e, there is no communication delay between the members of the network.

The state space representation of the first TEG is given by

$$\begin{aligned} x^{(1)}(k+1) &= \begin{pmatrix} \varepsilon & 8 & 9 \\ \mathrm{e} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} x^{(1)}(k) \oplus \begin{pmatrix} \varepsilon \\ \varepsilon \\ \mathrm{e} \end{pmatrix} u^{(1)}(k+1) , \\ y^{(1)}(k) &= \begin{pmatrix} \mathrm{e} & \varepsilon & \varepsilon \end{pmatrix} x^{(1)}(k) , \end{aligned}$$

where $\mathcal{G}(A^{(1)}, B^{(1)}, C^{(1)})$ is structurally controllable and structurally observable. The state space representations of the other TEGs are omitted due to lack of space.

In Figure 3 we can see the simulation results for the consensus variables of the network members, using the communication topology $\mathcal{G}_{comm}$. The dashed (black), dotted (blue), dashed-dotted line (green) and solid line (red) correspond to the consensus variables ξ_1, ξ_2, ξ_3 and ξ_4, respectively. Figure 3(a) depicts the consensus variables of the 4 members for the case that the initial conditions coincide with the eigenvector of $\mathfrak{A}$.

Obviously, the communication topology $\mathcal{G}_{comm}$ is strongly connected. Hence, a consensus is achieved. As stated in Theorem 2, a consensus can be achieved for arbitrary initial vectors (see Figure 3(b)). The consensus value is equal to the eigenvalue of $\mathfrak{A}$ which in this example is equal to the largest eigenvalue of all subsystems ($\eta = 7$). Hence, all the subsystems run as slow as the slowest subsystem 3, as k tends to ∞.

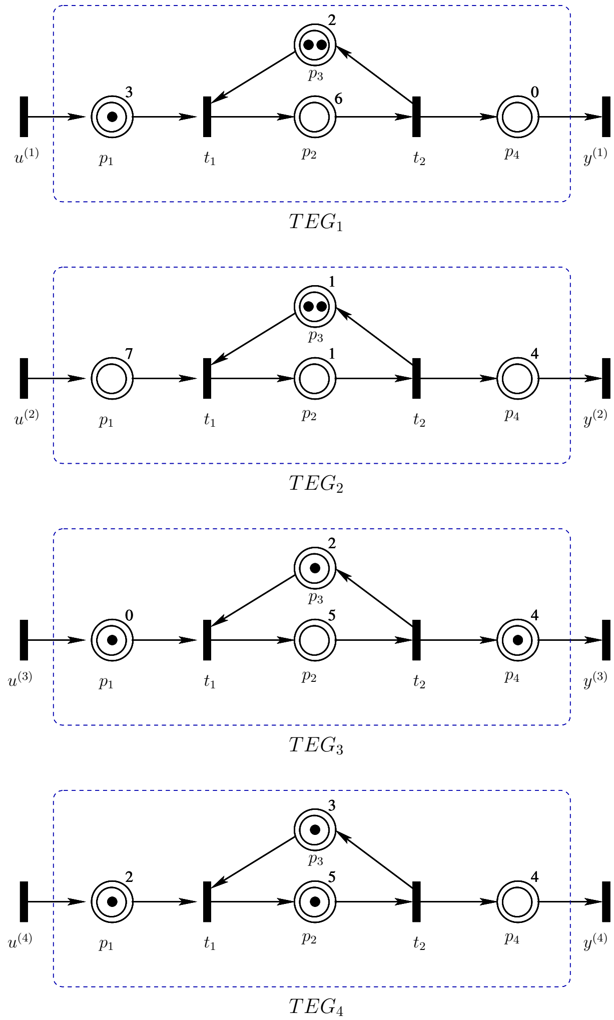

Fig. 1. TEGs supposed to get synchronized by a consensus feedback

VII. Conclusion

In this paper, we have investigated a consensus-based setup for stabilizing and synchronizing a class of timed event graphs. We introduced a framework, where a consensus-based control protocol in max-plus algebra is applied to a network consisting of structurally controllable and structurally observable TEGs. By transmitting output information to neighbours in the network, a consensus can be achieved regarding the asymptotic growth rates of the outputs of members. This results in a stable and, regarding to asymptotic growth rate of the outputs, synchronized closed loop system. A necessary and sufficient condition for the communication topology in order to achieve a consensus is provided. In future work, we investigate the case of switching communication topologies.

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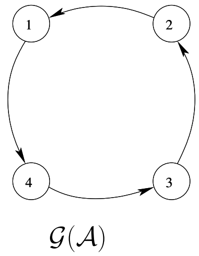

Fig. 2. Communication topology given by $\mathcal{A}$

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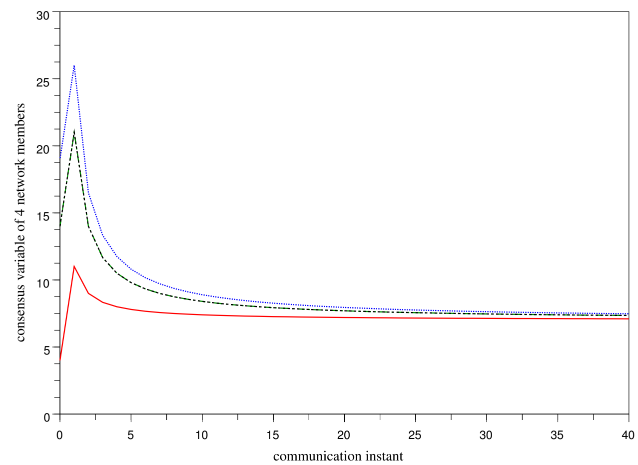
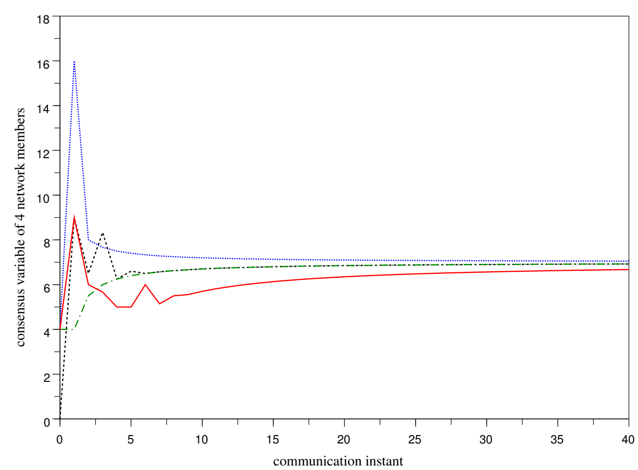


(a) $x(0)$: an eigenvector of $\mathfrak{A}$



(b) $x(0)$: a vector of e-elements

Fig. 3. Convergence to the asymptotic growth rate for the communication topology given by $\mathcal{G}(\mathcal{A})$ for different initial conditions