Optimal Control for Timed Event Graphs under Partial Synchronization

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***Abstract*— Timed event graphs (TEGs) are a subclass of timed Petri nets suitable to model decision-free timed discrete event systems. In classical TEGs, exact synchronization of two transitions T_1 and T_2 is available by requiring that transitions T_1 and T_2 fire simultaneously. In this paper, a new sort of synchronization, namely partial synchronization, is introduced: transition T_2 has to fire when transition T_1 fires, but transition T_1 is not influenced by transition T_2. Under some assumptions, optimal control, already available for classical TEGs, is extended to TEGs under partial synchronization.**

I. Introduction

Event graphs constitute a subclass of Petri nets where each place has exactly one upstream and one downstream transition and all arcs have weight 1. To capture time in event graphs, timed event graphs (TEGs) are built by equipping each place with a holding time (*i.e.,* duration a token must spend in a place before enabling the firing of the next transition). TEGs are decision-free: the interesting question is not which event happens next, but when the next possible events happen. Therefore, TEGs are suitable to model systems ruled by synchronization (*e.g.,* manufacturing processes, transport networks). It is a well-known fact that the timed/event behavior of a TEG, under the earliest functioning rule (*i.e.,* each transition with at least one upstream place fires as soon as it is enabled) can be expressed by linear relations over some dioids [1]. It leads to dynamic system representations and control methods built by analogy with classical control theory. For example, optimal control for TEGs is defined in [2], [3].

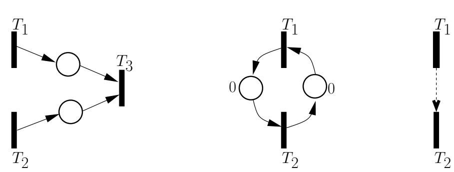

Fig. 1: Different sorts of synchronization for TEGs

For TEGs, synchronization classically refers to a requirement on the availability of resources (Fig. 1a): transition T_3 can fire *after* transition T_1 and transition T_2 fire. Some work has already been done to develop new types of synchronization and, therefore, to describe new phenomena. For example, in [4], soft synchronization is defined: a synchronization can be broken, but at a certain cost. A special case of synchronization is exact synchronization (Fig. 1b): transition T_1 and transition T_2 have to fire simultaneously. Partial synchronization (Fig. 1c) is a weak form of exact synchronization: transition T_2 can only fire *when* (not *after* as in Fig. 1a) transition T_1 fires and partial synchronization implies no condition on the firings of transition T_1.

Partial synchronization leads to a new class of timed discrete event systems: timed event graphs under partial synchronization (TEGPS). From a practical point of view, TEGPS are useful to model systems, divided in two decision-free subsystems: a main subsystem (transition T_1 in Fig. 1c) and a secondary subsystem (transition T_2 in Fig. 1c), such that the secondary subsystem has to adjust its behavior on the main subsystem, but the main subsystem is not affected by the secondary subsystem. For public transport networks, for example, a user (secondary subsystem) waits for its bus (main subsystem), but the bus does not wait for the user: the user has to adapt his behavior to the bus time-schedule.

In this paper, the behavior of the main subsystem is predetermined. Therefore, the main subsystem is represented by isolated transitions with known timed behaviors. Under this condition, the behavior of the decision-free secondary subsystem is investigated. In particular, the concept of optimal control is extended to TEGPS.

Necessary algebraic tools are presented in § II. In § III, TEG modeling over the dioid $\overline{\mathbb{Z}}_{max}$ is recalled. In § IV, partial synchronization is formally defined. The main contributions of this paper are introduced in § V: methods to determine the fastest output and the optimal input of the secondary subsystem under a predefined behavior of the main subsystem.

II. Algebraic Tools

The following is a short summary of basic results from dioid theory and residuation theory. The reader is invited to consult [1], [5] for more details.

A. Dioid Theory

A dioid $\mathcal{D}$ is a set endowed with two internal operations $\oplus$ (addition) and $\otimes$ (multiplication, often denoted by juxtaposition), both associative and both having a neutral element denoted ε and e respectively. Moreover, $\oplus$ is commutative and idempotent ($\forall a \in \mathcal{D}, a \oplus a = a$), $\otimes$ is distributive with

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respect to $\oplus$, and ε is absorbing for $\otimes$ ($\forall a \in \mathcal{D}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$).

The operation $\oplus$ induces an order relation $\preceq$ on $\mathcal{D}$, defined by $\forall a, b \in \mathcal{D}, a \succeq b \Leftrightarrow a \oplus b = a$. According to this order relation, $a \oplus b$ is the least upper bound of $\{a, b\}$.

A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. A complete dioid admits a greatest element $\top = \bigoplus_{x \in \mathcal{D}} x$. On a complete dioid, it is possible to define a new internal operation $\wedge$: $a \wedge b$ is the geatest lower bound of $\{a, b\}$. Clearly, $\wedge$ is idempotent and admits $\top$ as neutral element.

Example 1: A well-known complete dioid is the $(\max, +)$-algebra, denoted $\overline{\mathbb{Z}}_{max}$ (resp. $\overline{\mathbb{R}}_{max}$): $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ (resp. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$) is endowed with $\max$ as addition and $+$ as multiplication. ε is equal to $-\infty$ and $\top$ to $+\infty$. The associated order relation $\preceq$ is the usual order relation $\leq$.

By analogy with linear algebra, $\oplus$ and $\otimes$ are defined for matrices with entries in a dioid. Let $A, B \in \mathcal{D}^{n \times m}$ and $C \in \mathcal{D}^{m \times p}$:

- $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$
- $(A \otimes C)_{ij} = \bigoplus_{k=1}^{m} A_{ik} C_{kj}$

Furthermore, endowed with these operations, the set of square matrices with entries in a complete dioid is also a complete dioid.

The following theorem plays a fundamental role for the study of TEG behavior under the earliest functioning rule.

Theorem 1 (Kleene star theorem): The implicit inequality $x \succeq ax \oplus b$ defined over a complete dioid admits $x = a^* b$ as least solution with $a^* = \bigoplus_{i \geq 0} a^i$ (Kleene star). Besides, this solution achieves equality.

B. Residuation Theory

In ordered sets, like dioids, equation $f(x) = b$ may have either no solution, one solution, or multiple solutions. In order to give always a unique answer to the problem of mapping inversion, residuation theory [6], [7] provides, under some assumptions, the greatest solution (in accordance with the considered order) to the inequality $f(x) \preceq b$.

Definition 1 (Residuation): Let $f : \mathcal{E} \to \mathcal{F}$, with $(\mathcal{E}, \preceq)$ and $(\mathcal{F}, \preceq)$ ordered sets. An isotone (*i.e.*, order-preserving) mapping f is said to be residuated if for all $y \in \mathcal{F}$, the least upper bound of the subset $\{x \in \mathcal{E} | f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f.

The following theorem gives a very handy characterization of residuated mappings when the considered ordered sets are complete dioids.

Theorem 2 ([1]): Let $f : \mathcal{D}_1 \to \mathcal{D}_2$ be an isotone mapping defined over complete dioids. Mapping f is residuated if and only if $f(\varepsilon) = \varepsilon$ and, $\forall A \subseteq \mathcal{D}_1$, $f(\bigoplus_{x \in A} x) = \bigoplus_{x \in A} f(x)$.

Therefore, over complete dioids, $L_a : x \mapsto a \otimes x$ (left-product by a), respectively $R_a : x \mapsto x \otimes a$ (right-product by a), is residuated. Its residual is denoted by $L_a^\sharp(x) = a \backslash x$ (left-division by a), resp. $R_a^\sharp(x) = x / a$ (right-division by a). As left- and right-products are extended to matrices with entries in a complete dioid, left- and right-divisions are also extended to matrices with entries in a complete dioid. Besides, the calculation of left- and right-divisions in the matrix case can be derived from the calculation in the scalar case.

Proposition 1 ([1]): Consider a complete dioid $\mathcal{D}$, $A \in \mathcal{D}^{n \times m}$, $B \in \mathcal{D}^{n \times p}$, and $C \in \mathcal{D}^{p \times m}$, then:

- $B \backslash A \in \mathcal{D}^{p \times m}$ with $(B \backslash A)_{ij} = \bigwedge_{k=1}^{n} B_{ki} \backslash A_{kj}$
- $A / C \in \mathcal{D}^{n \times p}$ with $(A / C)_{ij} = \bigwedge_{k=1}^{m} A_{ik} / C_{jk}$

Residuation theory leads to a dual version of Th. 1.

Theorem 3: The implicit inequality $x \preceq a \backslash x \wedge b$ defined over a complete dioid admits $x = a^* \backslash b$ as greatest solution. Besides, this solution achieves equality.

Definition 2: A closure (resp. dual closure) mapping f is an isotone projection (*i.e.*, $f \circ f = f$) from a dioid $\mathcal{D}$ into itself, greater than (resp. less than) or equal to the identity mapping Id.

The next properties of closure mappings play a particular role in the following.

Lemma 1: Let f be a closure mapping. The least solution x of:

$$\begin{cases} x \succeq s \\ x \in \mathsf{Im}\,(f) \end{cases}$$

is $f\,(s)$.

Proof: As $f \succeq \mathsf{Id}$, $f\,(s)$ is a solution. Let x be a solution. As $x \in \mathsf{Im}\,(f)$, $x = f\,(x) \succeq s$. Besides, as f is isotone, $x = f\,(x) \succeq f\,(s)$. ■

Proposition 2 ([7],[8]): Let f be a residuated mapping. The following statements are equivalent:

- f is a closure mapping.
- $f^\sharp$ is a dual closure mapping.
- $f^\sharp = f \circ f^\sharp$
- $f = f^\sharp \circ f$

Lemma 2: Let f be a residuated closure mapping. The greatest solution x of:

$$\begin{cases} x \preceq s \\ x \in \mathsf{Im}\,(f) \end{cases}$$

is $f^\sharp\,(s)$.

Proof: $f^\sharp = f \circ f^\sharp$ implies $\mathsf{Im}\left(f^\sharp\right) \subseteq \mathsf{Im}\,(f)$, conversely $f = f^\sharp \circ f$ implies $\mathsf{Im}\,(f) \subseteq \mathsf{Im}\left(f^\sharp\right)$. Therefore, $\mathsf{Im}\,(f) = \mathsf{Im}\left(f^\sharp\right)$. The considered problem is equivalent to finding the greatest solution of:

$$\begin{cases} x \preceq s \\ x \in \mathsf{Im}\,(g) \end{cases}$$

with $g = f^\sharp$ a dual closure. By duality with Lem. 1, the greatest solution is $g\,(s) = f^\sharp\,(s)$. ■

III. TEG Description

This section is a short summary of TEG description over the dioid $\overline{\mathbb{Z}}_{max}$. The reader is invited to consult [1], [9] for more details.

For a timed Petri net, a dater t_i (*i.e.*, a non-decreasing function from $\mathbb{Z}$ to $\overline{\mathbb{Z}}_{max}$) is associated to each transition T_i. For $k \geq 0$, $t_i\,(k)$ is the time when firing k occurs. By convention, for $k < 0$, $t_i(k) = \varepsilon$ and for $k \geq 0$ $t_i(k) \in \mathbb{N}_0 \cup \{\varepsilon, \top\}$. From now on, only daters associated with transitions are considered: the previous convention always holds. For timed Petri nets, it is possible to partition the set of transitions into:

- input transitions $U_1, \ldots, U_p$: transitions without upstream places. The associated daters are denoted $u_1, \ldots, u_p$.
- output transitions $Y_1, \ldots, Y_q$: transitions without downstream places. The associated daters are denoted $y_1, \ldots, y_q$.
- state transitions $X_1, \ldots, X_n$: transitions with downstream and upstream places. The associated daters are denoted $x_1, \ldots, x_n$.

For TEGs with at most one token in each place, the state daters and output daters are solutions of $(\max, +)$-inequalities:

$$\left\{ \begin{array}{l} x(k) \succeq A_0 x(k) \oplus A_1 x(k-1) \oplus B u(k) \\ y(k) \succeq C x(k) \end{array} \right. \quad (1)$$

with A_0, A_1 in $\overline{\mathbb{Z}}_{max}^{n \times n}$, B in $\overline{\mathbb{Z}}_{max}^{n \times p}$, and C in $\overline{\mathbb{Z}}_{max}^{q \times n}$.

TEGs with places containing more than one token can be transformed into TEGs with at most one token in each place by adding state transitions.

From now on, only the behavior of TEGs (or TEGPS) under the earliest functioning rule (*i.e.*, each transition with at least one upstream place fires as soon as it is enabled) is investigated. In the following, two dual problems for TEGs, namely fastest output and optimal input, are briefly presented.

A. Fastest Output

The first problem is to calculate the fastest (*i.e.*, least) output $(y_u(k))_{0 \leq k \leq N}$ induced by a known input $(u(k))_{0 \leq k \leq N}$. The fastest output corresponds to the output under the earliest functioning rule.

First, Th. 1 transforms the state equation in (1) into an explicit recursive equation for the least state $(x_u(k))_{0 \leq k \leq N}$ induced by the input $(u(k))_{0 \leq k \leq N}$:

$$x_u(k) = A_0^* A_1 x_u(k-1) \oplus A_0^* B u(k)$$

with $x_u(-1) = \varepsilon$ by convention.

Second, the fastest output is directly derived from the least state:

$$y_u(k) = C x_u(k)$$

B. Optimal input

The second problem (see [2]) is to calculate the optimal (*i.e.*, greatest) input $(u_y(k))_{0 \leq k \leq N}$ ensuring, under the earliest functioning rule, a predetermined deadline $(y(k))_{0 \leq k \leq N}$.

First, the co-state $(\zeta(k))_{0 \leq k \leq N}$ (*i.e.*, the greatest state ensuring the deadline) is calculated. (1) leads to the following characterization of the co-state: it is the greatest solution of the implicit backward-recursive inequality:

$$\zeta(k) \preceq A_0 \backslash\!\!\circ\, \zeta(k) \wedge A_1 \backslash\!\!\circ\, \zeta(k+1) \wedge C \backslash\!\!\circ\, y(k) \quad (2)$$

with $\zeta(N+1) = \top$.

According to Th. 3, an explicit backward-recursive equation for the co-state is available:

$$\zeta(k) = (A_1 A_0^*) \backslash\!\!\circ\, \zeta(k+1) \wedge (C A_0^*) \backslash\!\!\circ\, y(k)$$

with $\zeta(N+1) = \top$.

Second, the optimal input is directly derived from the co-state:

$$u_y(k) = B \backslash\!\!\circ\, \zeta(k)$$

This can be interpreted as the "just-in-time" input to achieve a desired output.

IV. Partial Synchronization

In this section, the partial synchronization described in § I is formally defined.

Definition 3: Given a dater d, the partial synchronization mapping by d, denoted Φ_d, is a mapping from $\overline{\mathbb{Z}}_{max}$ to $\overline{\mathbb{Z}}_{max}$ defined by:

$$\Phi_d(x) = \bigwedge_{k' \in \mathbb{Z}} \{d(k') | d(k') \succeq x\}$$

By convention $\wedge \emptyset = \top$, this ensures the existence of $\Phi_d(x)$ for all daters d and all $x \in \overline{\mathbb{Z}}_{max}$.

Consider the system pictured in Fig. 1c, dater t_1 (resp. t_2) represents the behavior of transition T_1 (resp. T_2) when partial synchronization is neglected. Under partial synchronization, the behavior of transition T_1, the main subsystem, does not change. But the behavior of transition T_2, the secondary subsystem, is affected by the partial synchronization: each firing of transition T_2 is delayed until the next firing of transition T_1. The dater t_2^S, describing the behavior of transition T_2 under partial synchronization, is obtained from daters t_1 and t_2 using the partial synchronization mapping:

$$\begin{aligned} t_2^S(k) &= \Phi_{t_1}(t_2(k)) \\ &= \bigwedge_{k' \in \mathbb{Z}} \{t_1(k') | t_1(k') \succeq t_2(k)\} \end{aligned}$$

An equivalent definition of the partial synchronization mapping based on the image of d is:

$$\Phi_d(x) = \bigwedge \{x' \succeq x | x' \in \mathsf{Im}(d)\} \quad (3)$$

A direct extension of the partial synchronization mapping to the vector case is possible: let d a be a vector composed of n daters, Φ_d is a mapping from $\overline{\mathbb{Z}}_{max}^n$ to $\overline{\mathbb{Z}}_{max}^n$ defined by, for all $x \in \overline{\mathbb{Z}}_{max}^n$, $(\Phi_d(x))_i = \Phi_{d_i}(x_i)$.

The next propositions present some interesting properties of the partial synchronization mapping Φ_d in the scalar case. However an extension to vectors is obvious.

Proposition 3: Given a dater d, the partial synchronization mapping by d, Φ_d, is a closure mapping.

Proof: Φ_d is greater than or equal to the identity mapping, as:

$$\forall x \in \overline{\mathbb{Z}}_{max}, \Phi_d(x) = \bigwedge_{k' \in \mathbb{Z}} \{d(k') | d(k') \succeq x\} \succeq x$$

Φ_d is isotone:

$$\forall x_1, x_2 \in \overline{\mathbb{Z}}_{max}, x_1 \succeq x_2 \Rightarrow \Phi_d(x_1) \succeq \Phi_d(x_2)$$

The last part of the proof consists in showing that Φ_d is a projection. First:

$$\Phi_d(x) = \bigwedge_{k' \in \mathbb{Z}} \{d(k') | d(k') \succeq x\} \in \mathsf{Im}(d) \cup \{\top\}$$

If $\Phi_d(x) = \top$, $\Phi_d \circ \Phi_d(x) = \top$ as $\Phi \succeq \mathsf{Id}$.

If $\Phi_d(x) \in \mathsf{Im}(d)$, according to the definition of Φ_d in (3), $\Phi_d \circ \Phi_d(x) = \Phi_d(x)$. Consequently, Φ_d is a projection. ∎

In the following, we determine whether or not Φ_d is residuated. First, an auxiliary lemma on $\overline{\mathbb{Z}}_{max}$ is given.

Lemma 3: Consider $\mathcal{X} \subseteq \overline{\mathbb{Z}}_{max}$ and $\hat{x} = \bigoplus_{x \in \mathcal{X}} x$. Then, $\hat{x} \in \overline{\mathbb{Z}}_{max}$ and, besides, $\hat{x} \notin \mathcal{X} \Rightarrow \hat{x} = \top$.

Proof: As $\overline{\mathbb{Z}}_{max}$ is complete, $\hat{x} \in \overline{\mathbb{Z}}_{max}$. We will show that $\hat{x} \neq \top \Rightarrow \hat{x} \in \mathcal{X}$. If $\hat{x} = \varepsilon$, then obviously, $\forall x \in \mathcal{X}, x = \varepsilon$. Else, $\hat{x}$ is finite, by definition of $\hat{x}$, $\hat{x} - 1$ is not an upper bound of $\mathcal{X}$. Then, there exists $x' \in \mathcal{X}$ such that $\hat{x} \succeq x' \succ \hat{x} - 1$. Therefore, in $\overline{\mathbb{Z}}_{max}$, $x' = \hat{x}$. ∎

Proposition 4: Given a dater d, the partial synchronization mapping by d, Φ_d, is residuated.

Proof: $\Phi_d(\varepsilon) = \bigwedge_{k' \in \mathbb{Z}} \{d(k') | d(k') \succeq \varepsilon\} = \varepsilon$, as, by convention, $d(k') = \varepsilon$ for $k' < 0$. Consequently, according to Th. 2, it remains to show:

$$\forall \mathcal{X} \subseteq \overline{\mathbb{Z}}_{max}, \Phi_d\left(\bigoplus_{x \in \mathcal{X}} x\right) = \bigoplus_{x \in \mathcal{X}} \Phi_d(x)$$

By denoting $\hat{x} = \bigoplus_{x \in \mathcal{X}} x$ and as Φ_d is isotone, $\Phi_d(\hat{x}) \succeq \bigoplus_{x \in \mathcal{X}} \Phi_d(x)$.

Conversely, if $\hat{x} \in \mathcal{X}$, $\bigoplus_{x \in \mathcal{X}} \Phi_d(x) \succeq \Phi_d(\hat{x})$. Else, $\hat{x} \notin \mathcal{X}$, then, according to Lem. 3, $\hat{x} = \top$. As $\Phi_d \succeq \mathsf{Id}$, $\bigoplus_{x \in \mathcal{X}} \Phi_d(x) \succeq \bigoplus_{x \in \mathcal{X}} x = \hat{x} = \top$. Consequently, as $\top$ is the greatest element of $\overline{\mathbb{Z}}_{max}$, $\bigoplus_{x \in \mathcal{X}} \Phi_d(x) \succeq \Phi_d(\hat{x})$. ∎

Remark 1: The previous proposition does not hold anymore for $\overline{\mathbb{R}}_{max}$. Consider dater d and set $\mathcal{X}$ defined as:

$$d(k) = \begin{cases} \varepsilon \text{ if } k < 1 \\ 2 - \frac{1}{k} \text{ otherwise} \end{cases} \qquad \mathcal{X} = \left\{2 - \frac{1}{n} | n \geq 1\right\}$$

Then, $\Phi_d\left(\bigoplus_{x \in \mathcal{X}} x\right) = \Phi_d(2) = \top$, but $\bigoplus_{x \in \mathcal{X}} \Phi_d(x) = 2$. Then, according to Th. 2, Φ_d is not residuated.

Proposition 5: Consider a dater d, the residual of Φ_d is given by:

$$\Phi_d^\sharp(y) = \begin{cases} \top \text{ if } y = \top \\ \bigoplus_{k' \in \mathbb{Z}} \{d(k') | d(k') \preceq y\} \text{ otherwise} \end{cases}$$

Proof: $\Phi_d^\sharp(y)$ is the greatest solution x of $\Phi_d(x) \preceq y$ with $y \in \overline{\mathbb{Z}}_{max}$. If $y = \top$, $\forall x \in \overline{\mathbb{Z}}_{max}$, $\Phi_d(x) \preceq \top$. In particular, $\Phi_d(\top) \preceq \top$, then $\Phi_d^\sharp(\top) = \top$.

If $y \neq \top$, $\Phi_d(x) \preceq y$ is equivalent to, according to (3), $\exists x' \in \mathsf{Im}(d), x \preceq x' \preceq y$. Then:

$$\begin{aligned} \Phi_d^\sharp(y) &= \bigoplus \{x \in \overline{\mathbb{Z}}_{max}, \exists x' \in \mathsf{Im}(d) | x \preceq x' \preceq y\} \\ &= \bigoplus \{x' \in \mathsf{Im}(d) | x' \preceq y\} \text{ as } \mathsf{Im}(d) \subseteq \overline{\mathbb{Z}}_{max} \end{aligned}$$

∎

V. Behavior and Optimal Control of TEGPS

The following problem for TEGPS is considered. The behavior of the main subsystem is predefined: the main subsystem is represented by isolated transitions with known associated daters. The main objective of this section is to define the fastest output and the optimal input for the secondary subsystem.

From now on, it is assumed that only state transitions are subject to partial synchronization and that each state transition is synchronized by at most one dater. These conditions are not very restrictive. Indeed, it is possible to transform partial synchronization on input (or output) transitions into partial synchronization on state transitions by adding some state transitions. Moreover, as the partial synchronization mapping is entirely defined by the image of the synchronizing dater (see (3)), partial synchronization of a given transition by several daters $t_1, \ldots, t_r$ can be turned into a partial synchronization by a single dater $\tilde{t}$ such that $\mathsf{Im}(\tilde{t}) = \bigcap_{i=1}^{r} \mathsf{Im}(t_i)$.

Besides, to simplify the following discussion, it is also assumed that all state transitions are subject to partial synchronization. The state transitions without partial synchronization are synchronized by a dater t_e neutral with respect to partial synchronization:

$$t_e(k) = \begin{cases} \varepsilon \text{ if } k < 0 \\ k \text{ if } k \geq 0 \end{cases}$$

Let t be a vector of n daters such that state transition X_i is subject to partial synchronization by dater t_i. A state $x^S(k)$ is admissible for the secondary subsystem if and only if $x^S(k) \in \mathsf{Im}(\Phi_t)$ (*i.e.*, $\forall i \in [\![1, n]\!]$, $x_i^S(k) \in \mathsf{Im}(\Phi_{t_i})$). This new relation affects the behavior of the secondary subsystem. Therefore, the fastest output $\left(y_u^S(k)\right)_{0 \leq k \leq N}$ induced by the input $(u(k))_{0 \leq k \leq N}$ is the least solution of the following problem:

$$\begin{cases} x_u^S(k) \succeq A_0 x_u^S(k) \oplus A_1 x_u^S(k-1) \oplus Bu(k) \\ y_u^S(k) \succeq C x_u^S(k) \\ x_u^S(k) \in \mathsf{Im}(\Phi_t) \end{cases} \tag{4}$$

with $x_u^S(-1) = \varepsilon$.

A direct application of Th. 1 is not possible anymore due to the new condition: $x_u^S(k) \in \mathsf{Im}(\Phi_t)$.

A similar problem appears for the optimal input of the secondary subsystem. The optimal input $\left(u_y^S(k)\right)_{0 \leq k \leq N}$ ensuring the deadline $(y(k))_{0 \leq k \leq N}$ is the greatest solution of the following problem:

$$\begin{cases} \zeta^S(k) \preceq A_0 \backslash\!\!\!\circ\, \zeta^S(k) \wedge A_1 \backslash\!\!\!\circ\, \zeta^S(k+1) \wedge C \backslash\!\!\!\circ\, y(k) \\ u_y^S(k) \preceq B \backslash\!\!\!\circ\, \zeta^S(k) \\ \zeta^S(k) \in \mathsf{Im}(\Phi_t) \end{cases} \tag{5}$$

with $\zeta^S(N+1) = \top$.

In the following, a method is proposed to solve (4) and (5).

A. Triangular Representation of a TEG

A particular representation for TEGs is introduced to avoid implicit inequalities for the state in (4) and for the co-state in (5). This problem was already addressed in [1, §2.5.3].

Lemma 4 ([1]): A TEG is live (*i.e.*, each circuit has at least one token) if and only if there exists a permutation matrix P for which the matrix $P^{\top} A_0 P$ is strictly lower triangular.

Assuming that the TEG is live is not restrictive. Indeed, a TEG can always be represented by a live TEG with equivalent dynamics by:

- deleting the state transitions belonging to a circuit without tokens, but with a strictly positive holding time
- aggregating the state transitions belonging to a circuit without tokens, but with a holding time equal to 0. This case corresponds to the exact synchronization pictured in Fig. 1b for a circuit with only two transitions.

Therefore, in the following, it is assumed that A_0 is strictly lower triangular (*i.e.*, $(A_0)_{ij} = \varepsilon$ for $j \geq i$).

B. Fastest Output

In this section, the calculation of the fastest output $\left(y_u^S(k)\right)_{0 \leq k \leq N}$ induced by a predefined input $(u(k))_{0 \leq k \leq N}$ (formally, the least solution of (4)) is investigated. The state inequality is implicit:

$$x_u^S(k) \succeq A_0 x_u^S(k) \oplus A_1 x_u^S(k-1) \oplus B u(k)$$

However, as ε is absorbing for $\otimes$, the triangular structure of A_0 leads to a componentwise explicit inequality $x_{u,j}^S(k) \succeq \underline{r}_j(k)$ for $j \in [\![1, n]\!]$ with:

$$\underline{r}_j(k) = \bigoplus_{l=1}^{j-1} (A_0)_{jl}\, x_{u,l}^S(k) \oplus \bigoplus_{l=1}^{n} (A_1)_{jl}\, x_{u,l}^S(k-1) \oplus \bigoplus_{l=1}^{p} B_{jl} u_l(k)$$

Therefore, the problem of determining the least state induced by a predefined input $(u(k))_{0 \leq k \leq N}$ is equivalent to finding the least solution of:

$$\mathcal{P}_{jk} : \begin{cases} x_{u,j}^S(k) \succeq \underline{r}_j(k) \\ x_{u,j}^S(k) \in \mathsf{Im}\left(\Phi_{t_j}\right) \end{cases}$$

for all $j \in [\![1, n]\!]$ and $k \in [\![0, N]\!]$. Due to the structure of $\underline{r}_j(k)$, it is possible to consider the subproblems $\mathcal{P}_{jk}$ successively according to the following forward order: $\mathcal{P}_{10}, \ldots, \mathcal{P}_{n0}, \mathcal{P}_{11}, \ldots$ Besides, as Φ_{t_j} is a closure mapping (see Prop. 3), an expression of the least solution of each subproblem has already been presented in Lem. 1. Consequently, the least state is given by:

$$x_{u,j}^S(k) = \Phi_{t_j}\left(\underline{r}_j(k)\right)$$

for $j \in [\![1, n]\!]$ and $k \in [\![0, N]\!]$.

As before, the fastest output $\left(y_u^S(k)\right)_{0 \leq k \leq N}$ induced by a predefined input $(u(k))_{0 \leq k \leq N}$ is directly derived from the least state:

$$\forall k \in [\![0, N]\!], \quad y_u^S(k) = C x_u^S(k)$$

For fastest output calculation, considering the partial synchronization is equivalent to adding a supplementary condition $x(k) \in \mathsf{Im}(\Phi_t)$ to (1). Therefore, the least solution of (1) is less than or equal to the least solution of (4). Formally, for all k, $y_u(k) \preceq y_u^S(k)$: this corresponds to the intuitive fact that partial synchronization slows the system down.

C. Optimal Input

In this section, the calculation of the optimal input $\left(u_y^S(k)\right)_{0 \leq k \leq N}$ ensuring a predefined deadline $(y(k))_{0 \leq k \leq N}$ (formally, the greatest solution of (5)) is investigated. The co-state inequality is implicit:

$$\zeta^S(k) \preceq A_0 \backslash \zeta^S(k) \wedge A_1 \backslash \zeta^S(k+1) \wedge C \backslash y(k)$$

Using Prop. 1 leads to a componentwise inequality for the co-state:

$$\zeta_j^S(k) \preceq \bigwedge_{l=1}^{n} (A_0)_{lj} \backslash \zeta_l^S(k) \wedge \bigwedge_{l=1}^{n} (A_1)_{lj} \backslash \zeta_l^S(k+1) \wedge \bigwedge_{l=1}^{p} C_{lj} \backslash y_l(k)$$

As ε is absorbing for $\otimes$, $\varepsilon \otimes \top = \varepsilon$ and, therefore, for all z in $\overline{\mathbb{Z}}_{max}$, $\varepsilon \backslash z$ (*i.e.*, the greatest solution x of $\varepsilon \otimes x \preceq z$) is equal to $\top$. Consequently, as A_0 is strictly lower triangular:

$$\bigwedge_{l=1}^{n} (A_0)_{lj} \backslash \zeta_l^S(k) = \bigwedge_{l=j+1}^{n} (A_0)_{lj} \backslash \zeta_l^S(k)$$

Therefore, a componentwise explicit inequality is also obtained for the co-state. For $j \in [\![1, n]\!]$, $\zeta_j^S(k) \preceq \overline{r}_j(k)$ with:

$$\overline{r}_j(k) = \bigwedge_{l=j+1}^{n} (A_0)_{lj} \backslash \zeta_l^S(k) \wedge \bigwedge_{l=1}^{n} (A_1)_{lj} \backslash \zeta_l^S(k+1) \wedge \bigwedge_{l=1}^{p} C_{lj} \backslash y_l(k)$$

As before, the problem is divided into smaller subproblems: find the greatest solution $\zeta_j^S(k)$ of:

$$\mathcal{Q}_{jk} : \begin{cases} \zeta_j^S(k) \preceq \overline{r}_j(k) \\ \zeta_j^S(k) \in \mathsf{Im}\left(\Phi_{t_j}\right) \end{cases}$$

for all $j \in [\![1, n]\!]$ and $k \in [\![0, N]\!]$. Due to the structure of $\overline{r}_j(k)$, it is possible to consider the subproblems $\mathcal{Q}_{jk}$ successively according to the following backward order: $\mathcal{Q}_{nN}, \ldots, \mathcal{Q}_{0N}, \mathcal{Q}_{nN-1}, \ldots$ Besides, as Φ_{t_j} is a residuated closure mapping (see Prop. 3,4), an expression of the greatest solution of each subproblem has already been presented in Lem. 2. Consequently, the co-state is given by:

$$\zeta_j^S(k) = \Phi_{t_j}^{\sharp}\left(\overline{r}_j(k)\right)$$

for $j \in [\![1, n]\!]$ and $k \in [\![0, N]\!]$.

The co-state leads directly to the optimal input:

$$\forall k \in [\![0, N]\!], \quad u_y^S(k) = B \backslash \zeta^S(k)$$

As in the TEG case, the state induced by the optimal input is not necessarily equal to the co-state. However, it is always less than or equal to the co-state. Therefore, it ensures the deadline.

By analogy with fastest output calculation, a relation exists between the optimal output with or without partial synchronization: for all k, $u_y(k) \succeq u_y^S(k)$. Therefore, neglecting the partial synchronization for the optimal input calculation may lead to an output exceeding the deadline.

D. Application

The considered simple transport network is composed of two bus routes with transitions (stops) X_1, X_2, X_3 and X_4, X_5, X_6, such that transition X_2 (resp. transition X_5) is subject to partial synchronization with transition T_2 (resp. transition T_5) caused, for example, by a connection with a train line. The system is modelled by the TEG represented in Fig. 2. Dater t_2 (resp. dater t_5), representing the train time-schedule at transition T_2 (resp. transition T_5), takes the following values $(t_2(k))_{0\leq k\leq 5} = \{6, 15, 20, 31, 35, 42\}$ (resp. $(t_5(k))_{0\leq k\leq 5} = \{4, 11, 20, 26, 32, 42\}$).

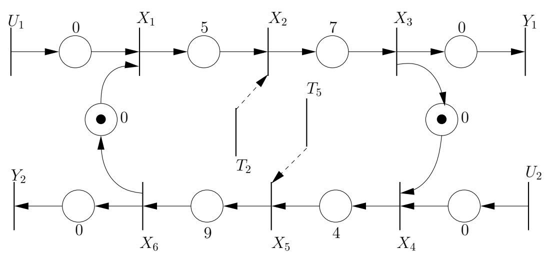

Fig. 2: A simple transport network

The matrices of the state-space model are:

$$A_0 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 9 & \varepsilon \end{pmatrix} \quad B = \begin{pmatrix} e & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & e \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}$$

$$A_1 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \quad C^\top = \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ e & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & e \end{pmatrix}$$

In this example, the matrix A_0 is already strictly lower triangular. Therefore, no permutation is necessary.

The following input u and deadline y are considered:

$$(u(k))_{0\leq k\leq 2} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 13 \\ 14 \end{pmatrix}, \begin{pmatrix} 25 \\ 27 \end{pmatrix} \right\}$$
$$(y(k))_{0\leq k\leq 2} = \left\{ \begin{pmatrix} 22 \\ 24 \end{pmatrix}, \begin{pmatrix} 43 \\ 38 \end{pmatrix}, \begin{pmatrix} 50 \\ 53 \end{pmatrix} \right\}$$

If the partial synchronization is neglected, the fastest output y_u induced by u and the optimal input u_y with respect to y are:

$$(y_u(k))_{0\leq k\leq 2} = \left\{ \begin{pmatrix} 12 \\ 13 \end{pmatrix}, \begin{pmatrix} 25 \\ 27 \end{pmatrix}, \begin{pmatrix} 39 \\ 40 \end{pmatrix} \right\}$$
$$(u_y(k))_{0\leq k\leq 2} = \left\{ \begin{pmatrix} 10 \\ 11 \end{pmatrix}, \begin{pmatrix} 28 \\ 25 \end{pmatrix}, \begin{pmatrix} 38 \\ 40 \end{pmatrix} \right\}$$

With partial synchronization, the following results are obtained:

$$\left(y_u^S(k)\right)_{0\leq k\leq 2} = \left\{ \begin{pmatrix} 13 \\ 13 \end{pmatrix}, \begin{pmatrix} 27 \\ 29 \end{pmatrix}, \begin{pmatrix} 42 \\ 41 \end{pmatrix} \right\}$$
$$\left(u_y^S(k)\right)_{0\leq k\leq 2} = \left\{ \begin{pmatrix} 10 \\ 7 \end{pmatrix}, \begin{pmatrix} 26 \\ 22 \end{pmatrix}, \begin{pmatrix} 37 \\ 38 \end{pmatrix} \right\}$$

As expected, for all k, $y_u^S(k) \succeq y_u(k)$ and $u_y^S(k) \preceq u_y(k)$.

VI. Conclusion

In this paper, a new sort of synchronization is formally introduced: partial synchronization. Under a predefined behavior of the main subsystem, an explicit expression for the state of the secondary subsystem is obtained. This expression allows us to constructively include partial synchronization in the dynamics of the secondary subsystem. Therefore, under some conditions, optimal control is defined for TEGPS. An interesting issue for future work is to adapt the operatorial representation of [5] to TEGPS.

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