Optimal Control of a Class of Timed Discrete Event Systems with Shared Resources, An Approach Based on the Hadamard Product of Series in Dioids.

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Abstract— **The topic of this paper is modeling and control of timed discrete event systems in a dioid framework if systems operate under the restriction of shared resources. The behavior of such systems can be elegantly modeled using the Hadamard product of series in dioids. Using residuation of the Hadamard product, it is possible to compute optimal control, where optimality is in the sense of a lexicographical order reflecting the chosen prioritization of subsystems. The paper concludes with an example, illustrating the efficiency of the proposed method.**

I. Introduction

Timed event graphs (TEGs) are a subclass of timed Petri nets where each place has exactly one upstream and one downstream transition and all arcs have weight 1. The time/event behavior of TEGs, under the earliest functioning rule (i.e., transitions fire as soon as they are enabled), can be expressed linearly over some dioids, e.g., the max-plus algebra [1]. TEGs can only model synchronization but not concurrency or choice. In many applications, like railway networks and manufacturing systems, there are only limited resources, which are shared among different users. For example, in a railway network, there may be single track segments which are used by multiple trains, but, at each instant of time, only one train can occupy the track. Hence, in order to model this conflict, we need to consider Petri nets where places may have more than one upstream and downstream transition. More precisely, we are focusing on the modeling of Resource Sharing (RS) phenomena. In the literature, various methods have been investigated to deal with the RS phenomenon. In [2], systems with RS are modeled by switching max-plus linear systems, where a system can switch between different modes of operation and in each mode, it is modeled by a linear max-plus system. Using model predictive control (MPC), the optimal switching sequence is obtained. In [3], the model consists of a TEG with some additional inequalities which model the limited availability of shared resources. In [4], conflicting timed event graphs are modeled in the max-plus algebra and an approach to calculate the cycle time is proposed. In [5], modeling and performance evaluation of timed Petri nets with different levels of priority are investigated. Three possible place/transition patterns are considered, namely, conflict, synchronization and priority configurations. In [6], we address modeling and control of systems with RS in the min-plus algebra, using a signal which denotes the number of resources available for each competing subsystem at each instant of time and model the overall system using this signal. In [7] some results on modeling and control of linear systems with additive inputs or outputs are presented. These results are based on the Hadamard product of series in dioids (see [8]).

In this paper, we use the idea presented in [7] to generalize the approach proposed in [3] and [6] to model systems with RS, and to compute the optimal control under a predefined priority policy. Optimality is in the sense of a lexicographical order reflecting the chosen prioritization of subsystems (users) sharing the resources. In essence, we are aiming at firing input transitions of subsystems as late as possible, while making sure that the firing of output transitions is not later than specified in given reference signals. Moreover, the control of lower-priority subsystems may not degrade the performance of higher-priority subsystems. This kind of prioritization can, e.g., be seen in emergency call centers, as considered in [5].

The paper is organized as follows. Section II recalls the necessary algebraic tools. In Section III, modeling of TEGs is revisited, and we propose a model for Petri nets with conflicts. Control of TEGs and systems subject to RS is addressed in Section IV, and Section V provides some conclusions.

II. Algebraic Preliminaries

The following is a summary of basic results from dioid theory and residuation theory. The interested reader is invited to peruse [1], [9] and [10] for more details.

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A. Dioid Theory

A dioid $\mathcal{D}$ is a set endowed with two internal operations denoted $\oplus$ (addition) and $\otimes$ (multiplication), both associative and having a neutral element denoted ε (zero element) and e (unit element), respectively. Moreover, $\oplus$ is commutative and idempotent ($\forall a \in \mathcal{D}, a \oplus a = a$), $\otimes$ distributes over $\oplus$, and ε is absorbing for $\otimes$ ($\forall a \in \mathcal{D}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). By convention, multiplication is often expressed by juxtaposition, i.e., $a \otimes b = ab$. The operation $\oplus$ induces an order relation $\preceq$ on $\mathcal{D}$, defined by: $\forall a, b \in \mathcal{D}, a \preceq b \Leftrightarrow a \oplus b = b$. A dioid is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. In this case, the greatest (in the sense of the above order) element of $\mathcal{D}$ is denoted $\top$ (the top element) and is equal to the sum of all its elements ($\top = \bigoplus_{x \in \mathcal{D}} x$). In a complete dioid, another binary operation ("greatest lower bound") denoted $\wedge$, can be defined by $a \wedge b = \bigoplus_{x \in \mathcal{D}_{ab}} x$ with $\mathcal{D}_{ab} = \{x \in \mathcal{D} | x \preceq a \quad \text{and} \quad x \preceq b\}$.

- The set $\overline{\mathbb{Z}}_{min} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the standard $\min$ operator as $\oplus$ and standard addition as $\otimes$ is a complete dioid, where $\varepsilon = +\infty$, $e = 0$ and $\top = -\infty$. Consequently, $\preceq$ on $\overline{\mathbb{Z}}_{min}$ is the reverse of the standard order (e.g., $3 \succeq 5$), and the greatest lower bound $\wedge$ is the standard $\max$ operator. Both operations $\oplus$ and $\otimes$ can be readily generalized for matrices of appropriate dimensions:

$$\forall A, B \in \overline{\mathbb{Z}}_{min}^{n \times m}, (A \oplus B)_{ij} = A_{ij} \oplus B_{ij},$$

$$\forall A \in \overline{\mathbb{Z}}_{min}^{n \times m}, B \in \overline{\mathbb{Z}}_{min}^{m \times p}, (A \otimes B)_{ij} = \bigoplus_{k=1}^{m} A_{ik} \otimes B_{kj}.$$

- Consider the set of formal power series in δ with exponents in $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$ and coefficients in $\overline{\mathbb{Z}}_{min}$, denoted $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$.
  - an element $s \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]$ can be interpreted as a map $s : \bar{\mathbb{Z}} \to \overline{\mathbb{Z}}_{min}$ and written as:

$$s = \bigoplus_{t \in \bar{\mathbb{Z}}} s(t)\delta^t,$$

    If there is only one t such that $s(t) \neq \varepsilon$, the series s is called a monomial and if there are a finite number of ts such that $s(t) \neq \varepsilon$, it is called a polynomial.
  - the set $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ can be equipped with operations $\oplus$ and $\otimes$ defined by: $\forall t \in \bar{\mathbb{Z}}, s_1, s_2 \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]$:

$$(s_1 \oplus s_2)(t) = s_1(t) \oplus s_2(t)$$

$$(s_1 \otimes s_2)(t) = \bigoplus_{j \in \bar{\mathbb{Z}}} s_1(j) \otimes s_2(t-j)$$

  - endowed with these operations, the set $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is a complete dioid with zero element $\varepsilon = \bigoplus_{t \in \bar{\mathbb{Z}}} \varepsilon \delta^t$ and unit element $e = \bigoplus_{t \in \bar{\mathbb{Z}}} e(t)\delta^t$, with

$$e(t) = \begin{cases} e, & t = 0 \\ \varepsilon & \text{otherwise.} \end{cases}$$

    The greatest lower bound of two series is given by:

$$(s_1 \wedge s_2)(t) = s_1(t) \wedge s_2(t).$$

    Note that $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ inherits its order from $\overline{\mathbb{Z}}_{min}$. From the general definition, $s_1 \preceq s_2 \Leftrightarrow s_1 \oplus s_2 = s_2$, it follows immediately that $s_1 \preceq s_2 \Leftrightarrow \forall t : s_1(t) \preceq s_2(t)$. Recall that the order here is inverted when compared to the standard order $\leqslant$.

- $s \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is nonincreasing if

$$s(t+1) \preceq s(t) \quad \forall t.$$

  The set of nonincreasing formal power series in $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is denoted by $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ and is a complete dioid. Because of nonincreasingness, elements in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$, can be represented compactly. E.g., the series $s = \bigoplus_{t \in \bar{\mathbb{Z}}} s(t)\delta^t$ with

$$s(t) = \begin{cases} e, & t \leqslant 0 \\ 2, & t = 1, 2 \\ 7, & t = 3, 4, 5 \\ 13, & t \geqslant 6 \end{cases}$$

  can be written as:

$$s = e\delta^0 \oplus 2\delta^2 \oplus 7\delta^5 \oplus 13\delta^{+\infty},$$

using the following rule between monomials:

$$n\delta^t \oplus n\delta^{t'} = n\delta^{max(t,t')}.$$

Moreover, the following also holds for monomials:

$$n\delta^t \oplus n'\delta^t = (n \oplus n')\delta^t = min(n, n')\delta^t.$$

Note that we use the same symbol to refer to multiplication in $\overline{\mathbb{Z}}_{min}$ and $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$. The same is true for addition, zero and unit elements.

Theorem 1 ([10]): Over a complete dioid $\mathcal{D}$, the implicit equation $x = ax \oplus b$ admits a least solution $x = a^*b$, where a^* is the Kleene star of a, defined by $a^* = \bigoplus_{i \in \mathbb{N}_0} a^i$ with $a^0 = e$, and $a^{i+1} = a \otimes a^i; i \in \mathbb{N}_0$.

B. Residuation Theory

Residuation theory (e.g., [11], [12]) provides, under some assumptions, the greatest solution (in accordance with the considered order) for the inequality $f(x) \preceq b$ where f is an isotone mapping (*i.e.* $a \preceq b \Rightarrow f(a) \preceq f(b)$) defined over ordered sets.

Definition 1: Let $f : \mathcal{D} \to \mathcal{C}$ be an isotone mapping with $(\mathcal{D}, \preceq)$ and $(\mathcal{C}, \preceq)$ being ordered sets. Mapping f is said to be residuated if, for all $y \in \mathcal{C}$, the greatest element of the subset $\{x \in \mathcal{D} | f(x) \preceq y\}$ exists and lies in this subset.

This element is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f. Mapping f is said to be dually residuated if, for all $y \in \mathcal{C}$, the smallest element of the subset $\{x \in \mathcal{D} | f(x) \succeq y\}$ exists and lies in this subset. This element is denoted $f^\flat(y)$, and mapping $f^\flat$ is called the dual residual of f.

Theorem 2: Let $f : \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ and $\mathcal{C}$ are complete dioids with their corresponding top (bottom) elements denoted $\top_{\mathcal{D}}(\varepsilon_{\mathcal{D}})$ and $\top_{\mathcal{C}}(\varepsilon_{\mathcal{C}})$, respectively. Mapping f is residuated iff $f(\varepsilon_{\mathcal{D}}) = \varepsilon_{\mathcal{C}}$ and $\forall \mathcal{A} \subseteq \mathcal{D}$ $f(\bigoplus_{x\in\mathcal{A}} x) = \bigoplus_{x\in\mathcal{A}} f(x)$. Mapping f is dually residuated iff $f(\top_{\mathcal{D}}) = \top_{\mathcal{C}}$ and $\forall \mathcal{A} \subseteq \mathcal{D}$ $f(\bigwedge_{x\in\mathcal{A}} x) = \bigwedge_{x\in\mathcal{A}} f(x)$.

Denote left multiplication by a in a dioid by L_a, i.e., $L_a : x \mapsto a \otimes x$. Mapping L_a is residuated. Its residual is denoted $L_a^\sharp : x \mapsto a \backslash\!\!\circ x$ and called *left division by* a. Therefore, $a \backslash\!\!\circ b$ is the greatest solution to inequality $a \otimes x \preceq b$ (*i.e.* $a \backslash\!\!\circ b = \bigoplus\{x \mid a \otimes x \preceq b\}$). Similarly, right multiplication by a, $R_a : x \mapsto x \otimes a$ is a residuated mapping. Its residual is denoted $R_a^\sharp : x \mapsto x /\!\!\circ a$ and called *right division by* a.
Residuation can be extended to the matrix case. Given the matrices $A \in \mathcal{D}^{m\times n}$ and $B \in \mathcal{D}^{m\times p}$, the greatest solution of $A \otimes X \preceq B$, with $\preceq$ understood elementwise, is given by $D = A \backslash\!\!\circ B$, where

$$D_{ij} = \bigwedge_{k=1}^{m} (A_{ki} \backslash\!\!\circ B_{kj}).$$

Theorem 3 ([13]): Let $f : \mathcal{D} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{D}$ be residuated mappings. The greatest solution of the equation $f(x) = g(x)$ such that $x \preceq b$ is obtained from the following algorithm where $\prod(x) = b \wedge x \wedge (g^\sharp \circ f)(x) \wedge (f^\sharp \circ g)(x)$:

Data: $\prod(x) = b \wedge x \wedge (g^\sharp \circ f)(x) \wedge (f^\sharp \circ g)(x)$
$m = 0$, $x_0 = b$;
repeat
  $x_{m+1} = \prod(x_m)$;
  $m = m + 1$;
until $x_{m+1} = x_m$;
Algorithm 1: The greatest $x_m \in \mathcal{D}$ such that $x_m = \prod(x_m)$

C. Hadamard Product

Definition 2: The Hadamard product of series in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ is defined as:

$$(s_1 \odot s_2)(t) = s_1(t) + s_2(t).$$

For this operation, the neutral element is $e_\odot = 0\delta^{+\infty}$, and the zero element $\varepsilon = +\infty\delta^{-\infty}$ is absorbing (i.e., $s \odot \varepsilon = \varepsilon \quad \forall s \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$).
The Hadamard product of series in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ is distributive over $\oplus$ and $\wedge$:

$$s_1 \odot (s_2 \oplus s_3) = (s_1 \odot s_2) \oplus (s_1 \odot s_3), \tag{1}$$

$$s_1 \odot (s_2 \wedge s_3) = (s_1 \odot s_2) \wedge (s_1 \odot s_3). \tag{2}$$

Proposition 1 ([7]): Mapping $\prod_a : x \mapsto a \odot x$ over $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ is residuated. Its residual is denoted $(\prod_a)^\sharp : x \mapsto a \odot^\sharp x$, where $(\prod_a)^\sharp(b)$ is the greatest series in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ which satisfies $a \odot x \preceq b$ for given a and b.
Between monomials, the following rules hold:

$$n\delta^t \odot n'\delta^{t'} = (n + n')\delta^{min(t,t')}$$

$$n\delta^t \odot^\sharp n'\delta^{t'} = \begin{cases} (n - n')\delta^t & \text{if} \quad t < t' \\ (n - n')\delta^{+\infty} & \text{else,} \end{cases}$$

For polynomials $p = \bigoplus_{i=1}^{m} n_i\delta^{t_i}$ and $p' = \bigoplus_{j=1}^{m'} n'_j\delta^{t'_j}$, these operations are defined as:

$$p \odot p' = \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{m'} n_i\delta^{t_i} \odot n'_j\delta^{t'_j} \tag{3}$$

$$p \odot^\sharp p' = \bigwedge_{j=1}^{m'}\bigoplus_{i=1}^{m} (n_i\delta^{t_i} \odot^\sharp n'_j\delta^{t'_j}) \tag{4}$$

Remark 1: Consider series s and s' in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$. Series s" defined by $s" : s"(t) = s(t) - s'(t)$ is not necessarily monotonic. The greatest monotonic series which is lower than or equal to s" is $s \odot^\sharp s'$.

Example 1: Consider two series $s = e\delta^0 \oplus 2\delta^2 \oplus 3\delta^4 \oplus 7\delta^{+\infty}$ and $s' = e\delta^1 \oplus 1\delta^3 \oplus 2\delta^8 \oplus 4\delta^{+\infty}$. Then, $s \odot^\sharp s' = e\delta^0 \oplus 2\delta^4 \oplus 5\delta^{+\infty}$. These series are shown in Fig.1. Furthermore, $(s \odot^\sharp s') \odot s' = e\delta^0 \oplus 2\delta^1 \oplus 3\delta^3 \oplus 4\delta^4 \oplus 7\delta^8 \oplus 9\delta^{+\infty} \preceq s$.

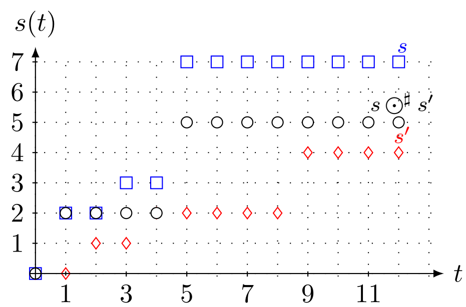

Fig. 1: Series $s = e\delta^0 \oplus 2\delta^2 \oplus 3\delta^4 \oplus 7\delta^{+\infty}$, $s' = e\delta^1 \oplus 1\delta^3 \oplus 2\delta^8 \oplus 4\delta^{+\infty}$, $s \odot^\sharp s' = e\delta^0 \oplus 2\delta^4 \oplus 5\delta^{+\infty}$.

III. Modeling

A. Modeling of TEG

Timed event graphs can be seen as linear systems in suitable semirings (e.g., [10], [1]). For instance, by associating to each transition $\mathbf{x}_i$ a "counter" function $x_i : \mathbb{Z} \to \overline{\mathbb{Z}}_{min}$, where $x_i(t)$ is equal to the number of firings of transition $\mathbf{x}_i$ before time t, it is possible to obtain a linear representation in $\overline{\mathbb{Z}}_{min}$.

A TEG operating under the earliest firing rule can be

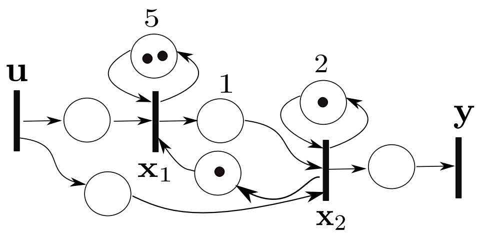

Fig. 2: A Single Input Single Output TEG [7].

modeled over $\overline{\mathbb{Z}}_{min}$ as:

$$\begin{cases} x(t) = A_0x(t) \oplus A_1x(t-1) \oplus ... \oplus A_Tx(t-T) \oplus \\ \qquad B_0u(t) \oplus ... \oplus B_Mu(t-M) \\ y(t) = C_0x(t) \oplus C_1x(t-1) \oplus ... \oplus C_Nx(t-N) \end{cases} \tag{5}$$

where $x(t) \in \overline{\mathbb{Z}}^n_{min}$, with n the number of internal transitions, $u(t) \in \overline{\mathbb{Z}}^m_{min}$, with m the number of input transitions, and $y(t) \in \overline{\mathbb{Z}}^l_{min}$, with l the number of output transitions. Input transitions are transitions without upstream places, output transitions are transitions without downstream places. All other transitions are called internal transitions. Matrices $A_0, ..., A_T$, $B_0, ..., B_M$ and $C_0, ..., C_N$ are of appropriate size with entries in $\overline{\mathbb{Z}}_{min}$. T is the maximum holding time of places between internal transitions, M denotes the maximum holding time of places connecting input transitions to internal ones, and N is the maximum holding time of places, connecting internal transitions to output ones.

The entries of vectors x, u, y can be represented by non-increasing formal power series, i.e., elements in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$, often referred to as δ-transforms, and the TEG model can be expressed in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ as:

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx, \end{cases} \tag{6}$$

where $x \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^n$, $u \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^m$ and $y \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^l$. Matrices A, B and C are of appropriate size with entries in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$.

According to Theorem 1, the least solution of (6) is $y = Gu$, where $G = CA^*B$ is referred to as the system transfer function matrix. The entries of G are periodic series in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ [1]. Moreover, the input to state transfer function matrix is given by $\tilde{H} = A^*B$.

A periodic series can be written as $s = p \oplus qr^*$ where $r = \nu\delta^\tau$, $p = \bigoplus_{i=0}^{n_p} n_i\delta^{t_i}$ is a polynomial representing a transient, $q = \bigoplus_{i=0}^{n_q} n_i\delta^{t_i}$ is a polynomial representing a pattern that is repeated every τ time units and after ν firings of the corresponding transition. The asymptotic slope $\sigma_\infty(s)$ of a periodic series is defined as $\sigma_\infty(s) = \frac{\nu}{\tau}$ and, in a manufacturing context, can be viewed as the production rate of the system. The asymptotic slope resulting from operations on periodical series s and s' is given by:

$$\sigma_\infty(s \oplus s') = min(\sigma_\infty(s), \sigma_\infty(s')),$$

$$\sigma_\infty(s \otimes s') = min(\sigma_\infty(s), \sigma_\infty(s')),$$

$$\sigma_\infty(s \odot s') = \sigma_\infty(s) + \sigma_\infty(s') = \frac{\nu\tau' + \nu'\tau}{\tau\tau'}.$$

If $\sigma_\infty(s) \leqslant \sigma_\infty(s')$ then $\sigma_\infty(s \odot^\sharp s') = \sigma_\infty(s) - \sigma_\infty(s')$[7].

Example 2: Consider the TEG shown in Fig.2, where we use the convention that holding times of places are 0 unless specified otherwise. Counters u, x_1 , x_2, and y are related as follows over $\overline{\mathbb{Z}}_{min}$:

$$\begin{cases} x_1(t) = 2 \otimes x_1(t-5) \oplus 1 \otimes x_2(t) \oplus u(t) \\ x_2(t) = x_1(t-1) \oplus 1 \otimes x_2(t-2) \oplus u(t) \\ y(t) = x_2(t). \end{cases}$$

Their respective δ-transforms are then related as:

$$\begin{cases} x_1 = 2\delta^5x_1 \oplus 1x_2 \oplus u \\ x_2 = \delta x_1 \oplus 1\delta^2x_2 \oplus u \\ y = x_2. \end{cases}$$

Consequently, by considering the state vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the following representation over $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$ is obtained :

$$\begin{aligned} x &= \begin{pmatrix} 2\delta^5 & 1 \\ \delta & 1\delta^2 \end{pmatrix} x \oplus \begin{pmatrix} e \\ e \end{pmatrix} u \\ y &= \begin{pmatrix} \varepsilon & e \end{pmatrix} x. \end{aligned} \tag{7}$$

Using Theorem 1, the transfer function of this TEG is then given by:

$$G = (\delta \oplus 1\delta^3)(2\delta^5)^*, \tag{8}$$

which is graphically represented in Fig. 3. Computation of the transfer function can be done using the software introduced in [14] and [15].

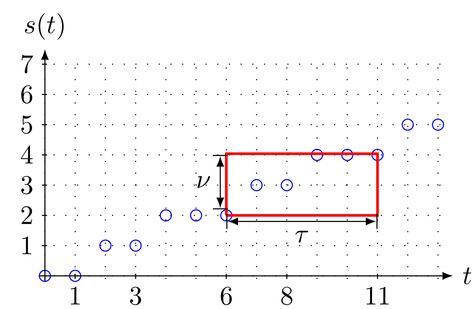

Fig. 3: Periodic series $G = (\delta^1 \oplus 1\delta^3)(2\delta^5)^*$.

B. Modeling Systems with Conflict

The Hadamard product can be used to model the behavior of systems with additive inputs (join) (see Fig. 4) or outputs (fork) (see Fig. 5). For the counter functions associated with the transitions in Fig. 4, at each time instant t, the following inequality must hold:

$$u_1(t) \odot u_2(t) \preceq y(t). \tag{9}$$

Considering the earliest firing rule, (9) turns to equality. Assume that the system output y and the input u_1 are known. In this case, it is interesting to find the greatest (in the sense of $\preceq$) u_2 such that (9) holds. This is given by :

$$u_2 = \bigoplus_{\{x|u_1 \odot x \preceq y\}} x = y \odot^{\sharp} u_1, \tag{10}$$

and u_2 represents the minimal (in the conventional order) number of firings of transition $\mathbf{u_2}$ to ensure (9).

For the counter functions associated with the transitions in Fig. 5, at each time instant t, the following inequality must hold:

$$u(t) \preceq y_1(t) \odot y_2(t). \tag{11}$$

Considering the earliest firing rule, (11) turns to equality. Assume that the system input u and the output y_1 are known. In this case, it is interesting to find the smallest (in the sense of $\preceq$) y_2 such that (11) holds. This is given by :

$$y_2 = \bigwedge_{\{x|y_1 \odot x \succeq u\}} x = u \odot^{\flat} y_1, \tag{12}$$

and y_2 represents the maximal (in the conventional order) number of firings of transition $\mathbf{y_2}$, given u and y_1.

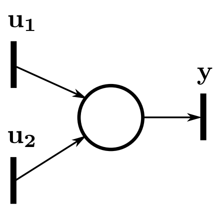

Fig. 4: System with additive inputs (join)

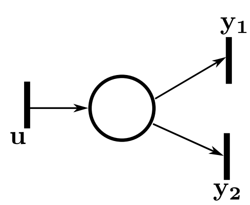

Fig. 5: System with additive outputs (fork)

IV. Optimal Control

A. Optimal Control of TEGs

A control input of a TEG is defined to be optimal, if it is the greatest input such that the resulting output y satisfies

$$y \preceq z, \tag{13}$$

where z is a given reference signal and "greatest" and $\preceq$ are to be interpreted in the sense of the order in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$. Hence, in conventional algebra, the optimal input corresponds to the least number of firings of the input transition that will ensure that the output transition fires at least $z(t)$ times (at any instant of time). This is also referred to as just-in-time control.

For the system modeled by (6), the following input-output relation holds:

$$y = CA^*Bu = Gu.$$

Hence, the optimal (just-in-time) control u^* is the greatest input u such that:

$$Gu \preceq z.$$

Due to Theorem 2, left multiplication is a residuated mapping, and

$$u^* = G \backslash\!\!\circ\, z. \tag{14}$$

The corresponding optimal behavior of the system is then obtained by:

$$\begin{cases} x^* = Ax^* \oplus Bu^* \\ y^* = Cx^* = Gu^*. \end{cases} \tag{15}$$

B. Optimal Control of Systems with RS Phenomena

We now consider an optimal control problem for a system with RS phenomena as shown in Fig. 6 where $G_i, i = 1, .., K$ and β are transfer functions of SISO TEGs. In this system, since there is conflict or choice, a priority policy is introduced. Without loss of generality, we assume that the places connecting input to internal transitions and internal transitions to output ones have zero holding times and no initial tokens.

Priority policy: In the following, we assume that the prior-

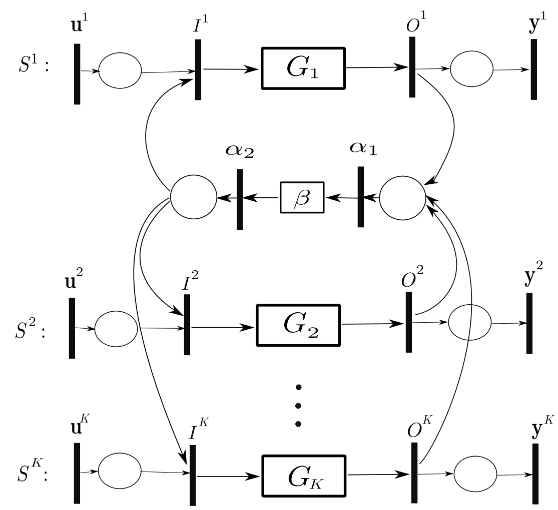

Fig. 6: A general system with RS phenomena.

ity of the system S^k is higher than S^{k+1}, for $k = 1, ..., K-1$, where K is the number of considered subsystems. The transfer function of subsystem S^k from input u^k to output y^k ignoring resource sharing with other subsystems (i.e., neglecting all the arcs going to the shared parts from other subsystems or vice versa) is denoted H_k.
First, the optimal control for subsystem S^1 is computed. Then the optimal control of system S^2 is computed under the restriction that the optimal behavior of S^1 is unchanged. This is repeated until the control for the lowest priority

subsystem is calculated. In other words, the control of each subsystem will not decrease the performance of the higher priority subsystems. This can be conveniently expressed using the following *lexicographic* order on $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^K$: For $u = (u^a, u^b), u' = (u'^a, u'^b) \in \overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]^K$

$$u \preceq_{\mathcal{L}} u' \Leftrightarrow \begin{cases} u^a \preceq u'^a, u^a \neq u'^a & \text{or} \\ u^a = u'^a \quad \text{and} \quad u^b \preceq u'^b \end{cases}, \tag{16}$$

where $\preceq$ is the previously introduced order in $\overline{\mathbb{Z}}_{min,\delta}[\![\delta]\!]$. Then, we look for the greatest control vector $u = (u^1, ..., u^K)$ (in the above lexicographic order) such that $y^k(t) \preceq z^k(t), k = 1, ..., K$.

This priority policy is often applied, for example, in emergency call centers. In these organizations there are different levels of emergency procedures. The priority is always given to the most urgent cases. That means, these cases are taken care of first. For the less urgent cases, the best is done, only after taking care of all the cases which are more urgent [5].

In order to apply this procedure for the system shown in Fig.6, first, the optimal control for S^1 is computed by neglecting all other subsystems. The optimal control u^{1*} and the corresponding x^{1*}, y^{1*} can be easily generated, using (14) and (15). In the next step, the optimal control for S^2 is calculated under the condition that the optimal behavior of S^1 is preserved. Since the priority of S^2 is greater than the one of S^k for $k = 3, ..., K$, in order to compute the optimal control of S^2, all the systems with lower priority are neglected.

For simplicity, first consider the system shown in Fig. 6 with only two subsystems S^1 and S^2. In the system, there are places with additive inputs and additive outputs. This can be modeled using the Hadamard product. According to (9) and (11), the following holds for the δ-transform of the respective counter function.

$$O^1 \odot O^2 \preceq \alpha_1, \tag{17}$$

$$\alpha_2 \preceq I^1 \odot I^2. \tag{18}$$

Due to the isotony of product $\otimes$, by multiplying (17) by β, we obtain:

$$\beta \otimes [O^1 \odot O^2] \preceq \beta \otimes \alpha_1 = \alpha_2. \tag{19}$$

(18) and (19) result in:

$$\beta \otimes [O^1 \odot O^2] \preceq I^1 \odot I^2. \tag{20}$$

The optimal control u^{1*} and the corresponding I^{1*} and O^{1*} can be computed directly from (14) and (15). In the next step, the optimal control for S^2 is computed, while I^{1*} and O^{1*} are preserved. Therefore, (20) in this case turns to:

$$\beta \otimes [O^{1*} \odot O^2] \preceq I^{1*} \odot I^2 \tag{21}$$

$$\Leftrightarrow \beta \otimes [O^{1*} \odot (G_2 \otimes I^2)] \preceq I^{1*} \odot I^2 \tag{22}$$

$$\Leftrightarrow O^{1*} \odot (G_2 \otimes I^2) \preceq \beta \backslash\!\!\!\circ (I^{1*} \odot I^2) \tag{23}$$

Data: number of subsystems K, series z^i, transfer relations $G_i, H_i, i = 1, ..., K$ and β
Result: $u^* = (u^{1*}, ...u^{K*})'$
$i = 1$;
$u^{1*} = H_1 \backslash\!\!\!\circ z^1$;
$I^{1*} = u^{1*}$;
$O^{1*} = G_1 \otimes I^{1*}$;
for $1 \leqslant i \leqslant K$ **do**
  $i = i + 1$;
  $n = 0$;
  $I^i_n = H_i \backslash\!\!\!\circ z^i$;
  $n = n + 1$;
  repeat
    $I^i_n = G_i \backslash\!\!\!\circ [(\beta \backslash\!\!\!\circ ((\bigodot_{j=1}^{i-1} I^{i*}) \odot I^i_{n-1})) \odot^\sharp (\bigodot_{j=1}^{i-1} O^{i*})] \wedge I^i_{n-1} \wedge (H_i \backslash\!\!\!\circ z^i)$;
  until $I^i_n = I^i_{n-1}$;
  $I^{*i} = I^i_n$;
  $O^{*i} = G_i \otimes I^{i*}$;
  $u^{*i} = I^{i*}$;
end

Algorithm 2: Method to compute the optimal control for the system shown in Fig. 6.

We now look for the greatest greatest u^2 (and therefore I^2) which satisfies (23) and $y^2 \preceq z^2$., Due to residuation of the Hadamard product, (23) is equivalent to:

$$G_2 \otimes I^2 \preceq (\beta \backslash\!\!\!\circ (I^{1*} \odot I^2)) \odot^\sharp O^{1*} \tag{24}$$

$$\Leftrightarrow I^2 \preceq G_2 \backslash\!\!\!\circ [(\beta \backslash\!\!\!\circ (I^{1*} \odot I^2)) \odot^\sharp O^{1*}] \tag{25}$$

$$\Leftrightarrow I^2 = (G_2 \backslash\!\!\!\circ [(\beta \backslash\!\!\!\circ (I^{1*} \odot I^2))) \odot^\sharp O^{1*}] \wedge I^2 \tag{26}$$

Due to Theorem 3, the problem of finding the greatest I^2 turns to finding the greatest fixed point of:

$$\Pi(I^2) = (G_2 \backslash\!\!\!\circ [(\beta \backslash\!\!\!\circ (I^{*1} \odot I^2))) \odot^\sharp O^{*1}] \wedge I^2 \wedge (H_2 \backslash\!\!\!\circ z^2). \tag{27}$$

According to Theorem 3, this greatest fixed point ($i.e., I^{2*}$) can be computed by the following recursive equation:

$$I^2_n = (G_2 \backslash\!\!\!\circ [(\beta \backslash\!\!\!\circ (I^{1*} \odot I^2_{n-1})) \odot^\sharp O^{1*}]) \wedge I^2_{n-1} \wedge (H_2 \backslash\!\!\!\circ z^2) \tag{28}$$

where $I^2_0 = H_2 \backslash\!\!\!\circ z^2$. Then $O^{2*} = G_2 I^{2*}$.

In the next step, the just-in-time control for S^k for $k = 3, ..., K$ can be computed. Using the same procedure for each S^k, neglecting all subsystems with lower priority and preserving the optimal behavior of all subsystems with higher priority, leads to the following control for $k = 3, ..., K$:

$$I^k_n = G_k \backslash\!\!\!\circ [(\beta \backslash\!\!\!\circ ((\bigodot_{i=1}^{k-1} I^{i*}) \odot I^k_{n-1})) \odot^\sharp (\bigodot_{j=1}^{k-1} O^{j*})] \wedge I^k_{n-1} \wedge (H_k \backslash\!\!\!\circ z^k), \tag{29}$$

where $I^k_0 = H_k \backslash\!\!\!\circ z^k$.

The whole procedure can be summarized as given in Algorithm 2.

In the case where β is only a monomial, $N\delta^T$, (i.e., the corresponding TEG is only one place with initially N tokens and a holding time T, Fig.6 represents a system with N shared resources. The above algorithm can also be extended to more general cases. For example, consider the Petri net structure shown in Fig. 7. Note that, H_1, respectively H_2, denotes the transfer function from u^1 to y^1, respectively u^2 to y^2, without considering the RS phenomenon (i.e., neglecting all the arcs going to the shared parts from the other subsystem or vice versa). Additionally, the corresponding input to state transfer function matrices are denoted $\tilde{H}_1$, respectively $\tilde{H}_2$. Similarly, for the system with the highest

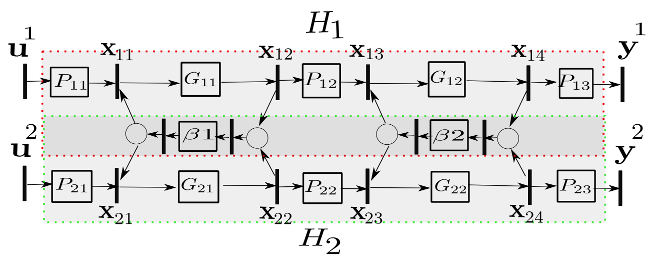

Fig. 7: A more general system with RS phenomenon.

priority, the optimal control input is computed, ignoring the RS phenomenon as $u^1_* = H_1 \circ\!\!\!\!\backslash z^1$. The corresponding states $(x^*_{11}, x^*_{12}, x^*_{12}, x^*_{12})^T$ can be computed using (15). Following the same reasoning as before, in order to compute the optimal input for system with the lower priority, preserving the optimal behavior of the higher priority system, the following inequalities must hold:

$$\begin{cases} \beta_1 \otimes [x^*_{12} \odot x_{22}] \preceq x^*_{11} \odot x_{21} \\ \beta_2 \otimes [x^*_{14} \odot x_{24}] \preceq x^*_{13} \odot x_{23} \end{cases}$$

which leads to:

$$\begin{aligned} \beta_1 \otimes [x^*_{12} \odot (G_{21} x_{21})] &\preceq x^*_{11} \odot x_{21} \\ \beta_2 \otimes [x^*_{14} \odot (G_{22} x_{23})] &\preceq x^*_{13} \odot x_{23}. \end{aligned} \tag{30}$$

Any x_{23} respecting (30) satisfies the restriction imposed by the shared part β_2 with the first system under optimal control. Therefore, we can replace x_{23} by $P_{22} G_{21} x_{21}$. This leads to:

$$\begin{aligned} &\beta_1 \otimes [x^*_{12} \odot (G_{21} x_{21})] \preceq x^*_{11} \odot x_{21} \\ &\beta_2 \otimes [x^*_{14} \odot (G_{22} P_{22} G_{21} x_{21})] \preceq x^*_{13} \odot (P_{22} G_{21} x_{21}). \end{aligned} \tag{31}$$

Hence, the following inequalities must hold:

$$\begin{aligned} x_{21} &\preceq \Pi_1(x_{21}) \\ x_{21} &\preceq \Pi_2(x_{21}) \end{aligned} \tag{32}$$

where

$$\Pi_1(x_{21}) = G_{21} \circ\!\!\!\!\backslash [(\beta_1 \circ\!\!\!\!\backslash (x^*_{11} \odot x_{21})) \odot^\sharp x^*_{12}]$$

and

$$\Pi_2(x_{21}) = (G_{22} P_{22} G_{21}) \circ\!\!\!\!\backslash [(\beta_2 \circ\!\!\!\!\backslash (x^*_{13} \odot (P_{22} G_{21} x_{21}))) \odot^\sharp x^*_{14}].$$

Therefore, the optimal x_{21} is the greatest solution of the following fixed point equation:

$$x_{21} = x_{21} \wedge \Pi_1(x_{21}) \wedge \Pi_2(x_{21}).$$

The method to compute the optimal control is summarized in Algorithm 3.

Data: series z^i, transfer relations $H_i, \tilde{H}_i, G_{ij}, P_{ik}$ and $\beta^i, j, i = 1, 2, k = 1, ..., 3$.
Result: $u^* = (u^{1*}, u^{2*})'$
$i = 1$;
$n = 0$;
$u^{1*} = H_1 \circ\!\!\!\!\backslash z^1$;
$(x^*_{11}, x^*_{12}, x^*_{12}, x^*_{12})^T = \tilde{H}_1 u^{1*}$;
$(u^2)^0 = H_2 \circ\!\!\!\!\backslash z^2$;
$(x^0_{21}, x^0_{22}, x^0_{23}, x^0_{24})^T = \tilde{H}_2 (u^2)^0$;
$n = n + 1$;
repeat
  $\Pi_1(x^{n-1}_{21}) = G_{21} \circ\!\!\!\!\backslash [(\beta_1 \circ\!\!\!\!\backslash (x^*_{11} \odot x^{n-1}_{21})) \odot^\sharp x^*_{12}]$;
  $\Pi_2(x^{n-1}_{21}) = (G_{22} P_{22} G_{21}) \circ\!\!\!\!\backslash [(\beta_2 \circ\!\!\!\!\backslash (x^*_{13} \odot (P_{22} G_{21} x^{n-1}_{21})))) \odot^\sharp x^*_{14}]$;
  $x^n_{21} = \Pi_1(x^{n-1}_{21}) \wedge \Pi_2(x^{n-1}_{21}) \wedge x^{n-1}_{21}$;
until $x^n_{21} = x^{n-1}_{21}$;
$x^*_{21} = x^n_{21}$;
$u^{2*} = P_{21} \circ\!\!\!\!\backslash x^*_{21}$;

Algorithm 3: Method to compute the optimal control for the system shown in Fig. 7.

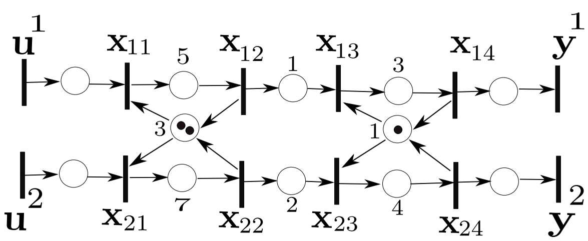

Fig. 8: An example for the system class shown in Fig. 7.

Example 3: Consider the system shown in Fig. 8. We apply Algorithm 3 to compute the optimal control input. The reference outputs are given by:

$$\begin{aligned} z^1 &= e\delta^{30} \oplus 1\delta^{31} \oplus 2\delta^{50} \oplus 3\delta^{65} \oplus 6\delta^{+\infty}, \\ z^2 &= e\delta^{52} \oplus 1\delta^{58} \oplus 2\delta^{64} \oplus 5\delta^{+\infty}. \end{aligned}$$

For this system,

$$\begin{aligned} &P_{11} = P_{21} = P_{13} = P_{23} = e\delta^0, P_{12} = e\delta^1, P_{22} = e\delta^2, \\ &G_{11} = e\delta^5, G_{12} = e\delta^3, G_{21} = e\delta^7, G_{22} = e\delta^4. \end{aligned}$$

where $A^1 = \begin{pmatrix} \varepsilon & 2\delta^3 & \varepsilon & \varepsilon \\ e\delta^5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e\delta^1 & \varepsilon & 1\delta^1 \\ \varepsilon & \varepsilon & e\delta^3 & \varepsilon \end{pmatrix}$, $B^1 = \begin{pmatrix} e\delta^0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}$ and $C^1 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & e\delta^0 \end{pmatrix}$ which leads to:

$$H_1 = C^1 A^{1*} B^1 = e\delta^9 (1\delta^4)^*$$

and

$$u^{1*} = H_1 \lparr z^1 = e\delta^{18} \oplus 1\delta^{22} \oplus 2\delta^{41} \oplus 3\delta^{48} \oplus 4\delta^{52} \oplus 5\delta^{56} \oplus 6\delta^{+\infty}.$$

As $(x^*_{11}, x^*_{12}, x^*_{13}, x^*_{14})^T = A^{1*}B^1u^{1*}$,

$$\begin{aligned} x^*_{11} &= \delta^{18} \oplus 1\delta^{22} \oplus 2\delta^{41} \oplus 3\delta^{48} \oplus 4\delta^{52} \oplus 5\delta^{56} \oplus 6\delta^{+\infty}, \\ x^*_{12} &= e\delta^{23} \oplus 1\delta^{27} \oplus 2\delta^{46} \oplus 3\delta^{53} \oplus 4\delta^{57} \oplus 5\delta^{61} \oplus 6\delta^{+\infty}, \\ x^*_{13} &= e\delta^{24} \oplus 1\delta^{28} \oplus 2\delta^{47} \oplus 3\delta^{54} \oplus 4\delta^{58} \oplus 5\delta^{62} \oplus 6\delta^{+\infty}, \\ x^*_{14} &= e\delta^{27} \oplus 1\delta^{31} \oplus 2\delta^{50} \oplus 3\delta^{57} \oplus 4\delta^{61} \oplus 5\delta^{65} \oplus 6\delta^{+\infty}. \end{aligned}$$

Similarly, $A^2 = \begin{pmatrix} \varepsilon & 2\delta^3 & \varepsilon & \varepsilon \\ e\delta^7 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e\delta^2 & \varepsilon & 1\delta^1 \\ \varepsilon & \varepsilon & e\delta^4 & \varepsilon \end{pmatrix}$, $B^2 = \begin{pmatrix} e\delta^0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}$ and $C^2 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & e\delta^0 \end{pmatrix}$ which leads to:

$$H_2 = C^2 A^{2*} B^2 = e\delta^{13}(1\delta^5)^*.$$

Therefore,

$$(u^2)^0 = H_2 \lparr z^2 = e\delta^{31} \oplus 1\delta^{36} \oplus 2\delta^{41} \oplus 3\delta^{46} \oplus 4\delta^{51} \oplus 5\delta^{+\infty},$$

and $(x^0_{21}, x_{22}p, x^0_{23}, x^0_{24})^T = A^{2*}B^2(u^2)^0$.
Initializing the recursive equation with

$$x^0_{21} = e\delta^{31} \oplus 1\delta^{36} \oplus 2\delta^{41} \oplus 3\delta^{46} \oplus 4\delta^{51} \oplus 5\delta^{+\infty}$$

results in:

$$u^{*2} = x^*_{21} = e\delta^0 \oplus 1\delta^5 \oplus 2\delta^{10} \oplus 3\delta^{28} \oplus 4\delta^{33} \oplus 5\delta^{+\infty}.$$

The resulting optimal output is:

$$\begin{aligned} y^{*1} &= e\delta^{27} \oplus 1\delta^{31} \oplus 2\delta^{50} \oplus 3\delta^{57} \oplus 4\delta^{61} \oplus 5\delta^{65} \\ &\quad \oplus 6\delta^{+\infty} \preceq z^1, \\ y^{*2} &= e\delta^{13} \oplus 1\delta^{18} \oplus 2\delta^{23} \oplus 3\delta^{41} \oplus 4\delta^{46} \oplus 5\delta^{+\infty} \preceq z^2. \end{aligned}$$

V. Conclusion

In this paper, we have discussed modeling and control of a class of timed Petri nets with resource sharing phenomena. In particular, several subsystems, each described by a timed event graph (TEG) compete for n resources modeled by tokens in a shared place. We assume a given prioritization policy. Under this policy, we provide optimal control (in the corresponding lexicographical order) for the overall system. In particular, the number of firings of the input transition of the top priority subsystem is, at any instant of time, as small as possible while guaranteeing that the output transition fires at least as often as specified by a given reference signal. The Hadamard product is used to model the RS phenomenon. Also, using its residuation, the optimal control under the given prioritization policy is computed. This approach is subsequently extended to a situation where multiple RS phenomena are present.

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