Optimal exploration and control for a robotic pick-up and delivery problem

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Abstract— **In this paper we address a problem where a robot moving on a line has to find and collect a finite number of objects and move them to a specified point. The robot is modeled as a second-order system and the task has to be completed in minimum time. Both the robot and the objects are represented by point masses. The objects are located at unknown places within a given interval and their pick-up and drop-off leads to a switching of the dynamics. The corresponding hybrid Optimal Control Problem (OCP) is investigated for the worst-case and a probabilistic case assuming a uniform distribution of the objects over the interval. We first derive optimal solutions for a single object. Then, we show that an optimal solution for the multi-object case consists of complete exploration followed by a deterministic optimal pick-up and drop-off (with possible intermediate drop-offs) of all objects. Thus, the computation of the exploration and the exploitation part of the control can be decoupled, similar to the single object case. The worst- and the probabilistic case optimal solutions are compared in a numerical example. The proposed methods are particularly relevant for different robotic applications like automated cleaning, search and rescue, harvesting, manufacturing etc.**

I. INTRODUCTION

Autonomous navigation under uncertainty is an inherent part of many robotic applications. Uncertainty emerges due to limited information of the environment or sensor disturbances and the motion trajectory needs to be adapted accordingly during mission execution [1]. Acquiring necessary information and achieving the overall goal are complementary subtasks that typically need to be accomplished under additional objectives, e.g. minimizing time or energy.

In this paper we address a time-OCP for a vehicle, modeled as a double integrator, that has to find, collect and move a finite number of objects back to the origin. The objects are located in a given interval on a line with non-negligible a-priori known masses. This represents a challenging theoretical problem for several reasons. First, it is well known that even linear deterministic state-to-state time-optimal motion allows for an analytic solution only in very few cases [2]. Second, an object pick-up and drop-off causes a switching of the robot's dynamics. Obtaining optimal solutions for such autonomous switched systems has an exponential combinatorial complexity that can be handled e.g. by two-stage iterative approximation schemes [3] or appropriate relaxations [4]. In addition, optimal exploration is an inherently complex problem in itself. Minimizing the expected time for detecting a target located on a real line according to a known probability distribution by a searcher, which can change the direction of its motion instantaneously, has a bounded maximal velocity and starts at the origin, has been initially addressed in [5], [6]. Even though this problem has received considerable attention from a game-theoretic perspective, e.g. [7], [8], [9], and in computer science [10], where it is also known as the 'cow-path problem', its solution for a general probability distribution is still an open question. The closely related problem of persistent monitoring of a limited one-dimensional mission space by a team of agents has been shown to effectively reduce to a parametric optimization problem [11]. The problem also resembles the elevator dispatching problem, for which receding horizon control approaches have been shown to outperform other methods in simulation [12]. Certainty equivalent event-triggered [13] and min-max optimization [14] schemes have also been employed to handle optimal control problems with uncertainties. The task of finding and tracking a moving target in a compact area has been formulated in a recursive Bayesian estimation framework [15]. While methods for Partially Observable Markov Decision Processes (POMDP's) can also be applied (e.g. [16]), they may become computationally infeasible for larger problem instants. To the best of the authors' knowledge, the optimal search problem has not yet been considered under the constraints (switching dynamics and bounded force) of the task at hand.

The first step towards an optimal combined exploration and control approach for a vehicle that has to find, collect and move objects in a 2-dimensional environment was based on a policy enforcing a time-optimal pick-up upon an object's detection, followed by a high-level certainty equivalent discrete optimization on a finite approximation of the motion in the environment [17]. These restrictions are omitted in this contribution, where the optimal exploration and control problem is formulated for a 1-dimensional mission space. Both the worst-case and a probabilistic case assuming uniform distribution of the objects are investigated in a hybrid manner. We first derive the optimal solutions for a single object. By exploiting the monotonicity of the cost function with respect to mass and distance, we show that the somewhat counterintuitive policy of complete exploration followed by a deterministic pick-up and drop-off of all objects is an optimal

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solution in the multi-object case. The proposed methods are compared in a numerical case study.

II. Problem formulation

Consider a finite number of objects $O = \{o_1, \ldots, o_L\}$, where every $o_l, l \in \{1, \ldots, L\}$ is uniquely characterized by its initial position $p^{(l)} \in [y_a, y_b] \subset \mathbb{R}_{\geq 0}$, and its mass $m^{(l)} \in \mathbb{R}_{\geq 0}$, i.e. $o_l = (p^{(l)}, m^{(l)})$. A robot has to find, collect and move all objects back to a depot, located at $y_d = 0$, in minimum time. Let the instantaneous detection of an object o_l be denoted by a discrete event $\delta_l(t^{(l)}) \in \Delta$ (a countable set) at time $t^{(l)} \in [0, T]$ with $y(t^{(l)}) = p^{(l)}$, where T is the free final time. This simple sensory paradigm covers practical setups where objects may be located alongside the motion line of the robot.

The system can be modeled by a hybrid automaton $\mathcal{H}$. Let the piecewise constant function $m_q : [0, T] \to M$ denote the overall mass of the robot that changes over time due to object pick-up and drop-off, taking values in the finite set $M = \{\nu \in \mathbb{R}_{\geq 0} | \nu = m_\emptyset + \sum_{o_l \in O} m^{(l)}\}$, where $m_\emptyset$ stands for the nominal mass of the robot. The discrete state $q(t) = (m_q(t), \mu_q(t))$ captures the current mass $m_q(t)$ corresponding to carrying a subset of objects $O_q \subseteq O$ and the piecewise constant function $\mu_q : [0, T] \to M \cup \{0\} = M_0$ denotes the mass of all objects $O_d \subseteq O$ that have been dropped off at the depot so far. As the co-domains of m_q and μ_q are finite, the discrete state takes values in a finite set $Q \subseteq M \times M_0$. The continuous state $x = [y\ v]' \in X$, where $y(t) \in \mathbb{R}_{\geq 0}$ is the current position of the robot and $v(t) \in \mathbb{R}$ its velocity, evolves according to a finite collection of vector fields $F = \{f_q\}_{q \in Q}$, i.e.

$$\dot{x}(t) = f_q(x(t), u(t)) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \frac{1}{m_q(t)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{1}$$

where $u : [0, T] \to [-1, 1] \subset \mathbb{R}$ is a piecewise continuous input. The overall time interval is divided into $L_s \in \mathbb{N}$ stages $[\tau_k, \tau_{k+1}), k \in \{0, \ldots, L_s - 1\}$ with $\tau_0 = 0$ and final interval $[\tau_{L_s - 1}, T]$, where τ_k corresponds to a discrete state switching resulting from an object pick-up or drop-off. The feasible discrete state transitions form a finite set $E \subseteq Q \times Q = E_1 \cup E_2$, where $e_1 \in E_1$ denotes the pick-up of object o_l with

$$e_1 = \{(i, j) | m_j = m_i + m^{(l)}\},$$

and $e_2 \in E_2$ represents the drop-off of all currently carried objects with

$$e_2 = \{(i, j) | m_j = m_\emptyset, \mu_j = \mu_i + m_j - m_\emptyset\}.$$

We assume that both pick-up and drop-off switchings occur only at zero velocity at the particular objects' locations autonomously, which can be captured by the invariant map $\mathrm{Inv} : Q \times \Delta \to 2^X$ with $O_f = O \setminus \{O_q, O_d\}$ and

$$\mathrm{Inv}(q, \delta_l) = \begin{cases} X \setminus \{[p^{(l)}\ 0]' | o_l \in O_f\}, & \text{if } O_q = \emptyset, \\ X \setminus \{\{[p^{(l)}\ 0]' | o_l \in O_f\}, [0\ 0]'\}, & \text{else,} \end{cases}$$

the guard map $G : E \times \Delta \to 2^X$ with

$$G(e, \delta_l) = \begin{cases} \{[p^{(l)}\ 0]'\}, & \text{if } e \in E_1, \\ \{[0\ 0]'\}, & \text{else;} \end{cases}$$

and the (trivial) reset map $R : E \times \Delta \times X \to X$ with $R(e, x) = x, \forall (e, x) \in E \times G(e)$. The occurrence of an event $\delta \in \Delta$ enables the corresponding transition $e = (q, q') \in E$, if $x \in G(e, \delta)$ or forces a transition out of q, if $x \notin \mathrm{Inv}(q, \delta)$. At the initial time $\tau_0 = 0$, let $\mathrm{Init} = \{(q(0), x(0))\} = \{((m_\emptyset, 0)', [0\ 0]')\}$. To support understanding, the reader is referred to Figure 1, which displays the hybrid automaton for two objects.

A feasible execution $\xi|_{[0,T]}$ of the hybrid automaton starts at Init and contains a sequence of strictly increasing initial, switching and final times $\tau = (\tau_0, \ldots, \tau_{L_s})$, a sequence of discrete states $(q(0), q(\tau_1), \ldots, q(\tau_{L_s}))$, a sequence of absolutely continuous state trajectories $x|_{[0,T]} = (x|_{[\tau_0, \tau_1)}, \ldots, x|_{[\tau_{L_s - 1}, \tau_{L_s}]})$ and a sequence of piecewise continuous controls $u|_{[0,T]} = (u|_{[\tau_0, \tau_1)}, \ldots, u|_{[\tau_{L_s - 1}, \tau_{L_s}]})$. Since an object drop-off is only possible at the depot, an execution will involve L pick-up events and up to L drop-off events, as it can be advantageous to collect several objects on the way and then drop them simultaneously off at the depot, such that $L_s \in \{L, \ldots, 2L\}$.

We want to solve the time-OCP for the worst (A) and a probabilistic (B) case in a receding horizon fashion, whenever new information about the objects' positions becomes available. In the latter, let the positions of the objects be independent identically distributed (iid) random variables with probability density functions (pdf) for all $l \in \{1, \ldots, L\}$

$$\mathcal{P}(p^{(l)}) = \begin{cases} \frac{1}{y_b - y_a}, & \text{if } y_a \leq p^{(l)} \leq y_b, \\ 0, & \text{else.} \end{cases} \tag{2}$$

Then, the two OCP's read as follows.

Problem 1. *Find the optimal execution continuation $\xi|^*_{[t,T]}$ of the hybrid automaton $\mathcal{H}$ given a past execution trajectory $\xi|_{[0,t]}$, such that*

$$\begin{aligned} &A) \min_u \max_{p^{(l)}, \forall o_l \in O_f} T, \\ &B) \min_u\ E\{T\}, \\ &\text{s.t. } q(T) = (m_\emptyset, (\sum_{o_l \in O} m^{(l)}))', x(T) = [0\ 0]'. \end{aligned}$$

The outline of the solution reads as follows. We first obtain necessary and sufficient optimality conditions for both OCP's for a single object scenario by employing results from classical time-optimal control for linear systems. Then, by exploiting properties of the system and the cost in the multi-object case, we show that an optimal solution for both OCP's is given by complete exploration followed by a deterministic optimal pick-up and drop-off.

III. Optimal control for a single object

Throughout this section, let $O = \{o_1\} = \{(p^{(1)}, m^{(1)})\}$ with an object pick-up carried out at time τ with $y(\tau) =$

$p^{(1)}$. For simplicity, assume that $y_a = 0$. The set of discrete states is $Q = \{q_{\emptyset,0} = (m_\emptyset, 0), q_{1,0} = (m_\emptyset + m^{(1)}, 0), q_{\emptyset,1} = (m_\emptyset, m^{(1)})\}$ such that

$$q(t) = \begin{cases} q_{\emptyset,0}, & t \in [0,\tau), \\ q_{1,0}, & t \in [\tau, T], \end{cases} \tag{3}$$
$$x(\tau) = [p^{(1)}\ 0]'.$$

and the continuous state evolves with unloaded $f_\emptyset$ and loaded f_1 dynamics. Note that $q(T) = q_{\emptyset,1}$ can be safely neglected in the following elaborations.

A. Preliminaries

The overall control in the first discrete state is $u_{\emptyset,0}|_{[0,\tau)} = (u_\emptyset^{(1)}, u_\emptyset^{(2)})$, where $u_\emptyset^{(i)}, i \in \{1,2\}$ are control trajectories corresponding to searching for the object and reaching the object with zero velocity upon its detection, respectively. Once the object is detected at time $t^{(1)}$ with $v(t^{(1)}) \geq 0$, it can be reached at time τ with $v(\tau) = 0$ by employing a classical time-optimal bang-bang controller with a single switching after a time t_2 [2], given by

$$u_\emptyset^{(2)}(t) = \begin{cases} -1, & t \in [t^{(1)}, t^{(1)} + t_2), \\ 1, & t \in [t^{(1)} + t_2, \tau). \end{cases} \tag{4}$$

To obtain a standard representation, we introduce an additional state for the time, leading to an extended system state $\tilde{x} = [y\ v\ t^{(1)}]'$ with $\dot{\tilde{x}}_3 = 1$. Solving the ODE for $m_q = m_\emptyset$ and (4), and applying the boundary condition for the object's pick-up $y(t^{(1)}) = y(\tau) = p^{(1)}$, yields the cost in the unloaded discrete state

$$\ell_1(q_{\emptyset,0}, \tilde{x}) = \tau = \tilde{x}_3 + \underbrace{(1+\sqrt{2})}_{c} m_\emptyset \tilde{x}_2.$$

Reaching the object with zero velocity allows for its pick-up, causing a switch to the loaded discrete state. Then, a bang-bang controller of the form

$$u_{1,0}(t) = \begin{cases} -1, & t \in [\tau, \tau + t_3), \\ 1, & t \in [\tau + t_3, T], \end{cases} \tag{5}$$

yields the time-optimal motion to the depot in time

$$\ell_2(q_{1,0}, p^{(1)}) = 2\sqrt{m_1 p^{(1)}}.$$

Hence, the overall time for finding, picking up and moving the object back to the origin is given by

$$T = J(q, \tilde{x}) = \ell_1(q_{\emptyset,0}, \tilde{x}) + \ell_2(q_{1,0}, p^{(1)}). \tag{6}$$

B. Worst-case solution

The (certainty equivalent) worst-case intuitively corresponds to the object being located the furthest away from the origin, i.e. $p^{(1)} = y_b$. The optimal control is given by $u_\emptyset^{(1)} = (1|_{[0,\sqrt{y_b m_\emptyset})}, -1|_{[\sqrt{y_b m_\emptyset}, t^{(1)})})$, followed by (4) and then (5) after the pick-up, with a predicted worst-case time

$$T_{\text{wc}} = 2(\sqrt{y_b m_\emptyset} + \sqrt{y_b m_1}). \tag{7}$$

Once the object is detected at time $t^{(1)}$, the actual optimal control solving the task can be computed deterministically.

C. Probabilistic solution

As the object's location is assumed to be uniformly distributed, the cost in the probabilistic case is given by the expected value of (6), i.e.

$$E\{T\} = \underbrace{\frac{1}{y_b}\int_0^{y_b} \ell_1(q_{\emptyset,0}, \tilde{x}) d\tilde{x}_1}_{E\{\ell_1\}} + \underbrace{\frac{1}{y_b}\int_0^{y_b} \ell_2(q_{1,0}, p^{(1)}) dp^{(1)}}_{E\{\ell_2\}}.$$

The second integral can be easily evaluated for a single object and yields an additive constant in the cost that can be neglected for now. Substituting the relation $d\tilde{x}_1 = \tilde{x}_2 dt$, we equivalently obtain for the first integral

$$E\{\ell_1\} = \frac{1}{y_b}\int_0^{\tau} (\tilde{x}_2\tilde{x}_3 + c m_\emptyset \tilde{x}_2^2) dt, \tag{8}$$

where τ is free and the constraints

$$\tilde{x}(0) = [0\ 0\ 0]',\ \tilde{x}_1(\tau) = y_b,\ \tilde{x}_2(\tau) = 0$$

must be satisfied. Thus, we have transformed the probabilistic partial OCP in the unloaded mode into a nonlinear OCP with free final time for which we can derive necessary optimality conditions.

Let the control Hamiltonian be given by

$$H(\tilde{x}, \lambda, u_\emptyset^{(1)}) = \frac{1}{y_b}(\tilde{x}_2\tilde{x}_3 + c m_\emptyset \tilde{x}_2^2) + \lambda' \begin{bmatrix} \tilde{x}_2 & \frac{u_\emptyset^{(1)}}{m_\emptyset} & 1 \end{bmatrix}', \tag{9}$$

where $\lambda(t) \in \mathbb{R}^3$ is the costate vector with absolutely continuous dynamics

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial \tilde{x}} = -\begin{bmatrix} 0 & \frac{1}{y_b}(2cm_\emptyset\tilde{x}_2 + \tilde{x}_3) + \lambda_1 & \frac{1}{y_b}\tilde{x}_2 \end{bmatrix}'. \tag{10}$$

Applying Pontryagin's minimum principle (PMP), $\exists$ an optimal state $\tilde{x}^*$, a control $u_\emptyset^{(1)*}$, and a nontrivial costate λ^* trajectory, such that $\forall t \in [0,\tau)$,

$$H(\tilde{x}^*(t), \lambda^*(t), u_\emptyset^{(1)*}(t)) = 0 = \min_{u_\emptyset^{(1)}(t)} H(\tilde{x}(t), \lambda(t), u_\emptyset^{(1)}(t)). \tag{11}$$

Theorem 1. *The optimal control is $u_\emptyset^{(1)*}(t) \in \{-1, -\frac{1}{2c}, 1\}$ for $t \in [0,\tau)$.*

Proof. From (9) and (11), $\lambda_2^*(t) u_\emptyset^{(1)*}(t) \leq \lambda_2^*(t) u_\emptyset^{(1)}$ and for $\lambda_2(t) \neq 0, t \in [0,\tau)$, we obtain $u_\emptyset^{(1)}(t) = -\text{sign}(\lambda_2(t))$, i.e. $u_\emptyset^{(1)} \in \{\pm 1\}$. As the input is bounded, $\lambda_2(t) = 0$ can hold for isolated times, without violating the PMP. If $\lambda_2(t) = 0$ over an interval, i.e. $\forall t \in [t_1, t_2] \subset [0,\tau)$, also $\dot{\lambda}_1(t) = 0$ and $\lambda_2(t) = 0$ must hold. As $H(\tilde{x}^*(t), \lambda^*(t), u_\emptyset^{(1)*}(t)) = 0$, also $\lambda_1(t) = 0$ and $\lambda_3(t) = 0$, leading to another possible input value $u_\emptyset^{(1)}(t) = -\frac{1}{2c}$. □

Now we want to derive sufficient conditions for optimality.

Lemma 1. *The optimal control with $\tilde{t}_1 \in [0,\tau)$ is*

$$u_\emptyset^{(1)*}(t) = \begin{cases} 1, & t \in [0, \tilde{t}_1), \\ -\frac{1}{2c}, & t \in [\tilde{t}_1, \tau). \end{cases} \tag{12}$$

Proof. From (9) and $\tilde{x}_1 \in \mathbb{R}_{\geq 0}$, $\lambda_2(0) < 0$ holds and the optimal control must start with $u_\emptyset^{(1)} = 1$. Assuming that $\lambda_2(0) = a \in \mathbb{R}_{<0}$ for the interval $[0, \tilde{t}_1]$, the adjoint variable is $\lambda_2(t) = a - \frac{2c+1}{2} t^{(2)} - \lambda_1 t$ by integration of (10). If $\lambda_2(\tilde{t}_1) = \dot{\lambda}_2(\tilde{t}_1) = 0$, $\lambda_1 = -\sqrt{2(1+2c)a}$, the set of possible bang-bang sequences are $\{(1), (1, -1), (1, -1/2c), (1, -1/2c, -1)\}$, or with $\tau = \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3$ in generalized form

$$u_\emptyset^{(1)}(t) = \begin{cases} 1, & t \in [0, \tilde{t}_1), \\ -\frac{1}{2c}, & t \in [\tilde{t}_1, \tilde{t}_1 + \tilde{t}_2), \\ -1, & t \in [\tilde{t}_1 + \tilde{t}_2, \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3). \end{cases} \tag{13}$$

Integrating (1) with $m_q = m_\emptyset$ and (13), using the switching and final conditions $x_1(\tau) = y_b, x_2(\tau) = v_c$ and analyzing the expressions for $\tilde{t}_3 = 0$ and $\tilde{t}_2 = 0$, we obtain the boundaries

$$\sqrt{\frac{2(m_\emptyset y_b + c m_\emptyset{}^2 v_c{}^2)}{2c-1}} \leq \tilde{t}_1 \leq \sqrt{m_\emptyset y_b + \frac{m_\emptyset^2 v_c^2}{2}},$$

for $0 \leq v_c \leq \sqrt{\frac{2y_b}{m_\emptyset}}$. By substituting the solutions of the system equations, the cost $E\{\ell_1\}$ becomes a function of $\tilde{t}_1$ and v_c. Evaluating $\frac{\partial E\{\ell_1\}}{\partial \tilde{t}_1}$ yields the potential extrema $\tilde{t}_1 = \left\{\sqrt{\frac{2m_\emptyset y_b}{2c-1}}, \sqrt{m_\emptyset y_b + \frac{m_\emptyset^2 v_c^2}{2}}\right\}$, and by employing the derived bounds and the second derivative $\frac{\partial^2 E\{\ell_1\}}{\partial \tilde{t}_1^2}$, we can verify that the first value corresponds to the minimum. By substituting the result for $\tilde{t}_1$ in $E\{\ell_1\}$ and evaluating $\frac{\partial \ell_1}{\partial v_c}$, the potential final velocity is $v_c = \left\{0, \sqrt{\frac{2y_b}{m_\emptyset}}\right\}$. Analyzing $\frac{\partial^2 E\{\ell_1\}}{\partial v_c^2}$ yields $v_c^* = 0$ and we obtain $\tilde{t}_2 = 2c\sqrt{\frac{2m_\emptyset y_b}{2c-1}}$, accordingly. Thus, the minimum is given for $\tilde{t}_3 = 0$ and the optimal bang-bang controller is (12). □

The expected cost in the unloaded and the loaded discrete states can be obtained by direct integration.

IV. Optimal control for multiple objects

In general, due to the dependence of the solution on the unknown object positions, i.e. of the trajectories on the unknown detection times $t^{(l)}$, all possible optimal executions need to be evaluated to find the optimal solution in the multi-object case. However, by exploiting properties of the problem at hand, we will show that there exists an optimal execution, whose switching and detection times satisfy an order relation.

The cost of an execution in the multi-object case is given by the sum of the times for moving between pairs of states $x(\tau_k), x(\tau_{k+1}) \in X, k \in \{0, \ldots, L_s - 1\}$ with continuous dynamics f_k corresponding to the current discrete state $k = (m_k, \mu_k) \in Q$, where τ_k, τ_{k+1} denote discrete state transition instants, i.e.

$$J(\xi|_{[0,T]}) = \tau_{L_s} = \sum_{k=0}^{L_s-1} (\tau_{k+1} - \tau_k). \tag{14}$$

Intuitively, upon the occurrence of the detection event δ_l at time $t^{(l)}$, the robot can choose either to immediately pick up o_l or continue exploration. Upon picking up o_l, the robot can go back to the depot to drop-off all currently carried objects or continue exploration. Based on these binary options, we will show that an execution consisting of complete exploration followed by a deterministic optimal pick-up and drop-off (with possible intermediate drop-offs) of all objects constitutes an optimal solution for the multi-object case, as stated in the following theorem.

Theorem 2. *An optimal execution $\xi|_{[0,T]}^*$ of $\mathcal{H}$ for $O = \{o_1, \ldots, o_L\}$ solving Problem 1 satisfies*

$$t^{(l)} \leq \tau_1, \forall l \in \{1, \ldots, L\}, \tag{15}$$

where τ_1 is the time instant of the first pick-up of any object.

Proof. Assume first that $O' = \{o_1, o_2\}$ and let $\delta_l(t^1)$ occur first with a past execution $\xi|_{[0,t^1]}$. The corresponding hybrid automaton $\mathcal{H}$ capturing all possible executions is depicted in Figure 1. As every $(\tau_{k+1} - \tau_k)$ in (14) is monotonically increasing with the mass m_k and the distance $\|x_2(\tau_{k+1}) - x_1(\tau_k)\|$, $m_{12} \geq m_1 \geq m_\emptyset, m_{12} \geq m_2 \geq m_\emptyset$ and $t^{(1)} \leq t^{(2)}$ implies that $p^{(1)} \leq p^{(2)}$, we can see that

$$\begin{aligned} \tau_3^{10} &= \underbrace{(\tau_1^1 - 0)}_{\propto m_\emptyset, p^{(1)}} + \underbrace{(\tau_2^2 - \tau_1^1)}_{\propto m_1, |p^{(2)} - p^{(1)}|} + \underbrace{(\tau_3^{10} - \tau_2^2)}_{\propto m_{12}, p^{(2)}} \\ \geq \tau_3^{20} &= \underbrace{(\tau_1^2 - 0)}_{\propto m_\emptyset, p^{(2)}} + \underbrace{(\tau_2^1 - \tau_1^2)}_{\propto m_2, |p^{(2)} - p^{(1)}|} + \underbrace{(\tau_3^{20} - \tau_2^1)}_{\propto m_{12}, p^{(1)}} \end{aligned}$$

holds for any limited input u, as the former includes the cost for moving along a longer segment with a higher mass. Further,

$$\begin{aligned} \tau_4 &= \underbrace{(\tau_1^1 - 0)}_{\propto m_\emptyset, p^{(1)}} + \underbrace{(\tau_2^0 - \tau_1^1)}_{\propto m_1, p^{(1)}} + \underbrace{(\tau_3^2 - \tau_2^0)}_{\propto m_\emptyset, p^{(2)}} + \underbrace{(\tau_4 - \tau_3^2)}_{\propto m_2, p^{(2)}} \\ = \tau_4' &= \underbrace{(\tau_1^2 - 0)}_{\propto m_\emptyset, p^{(2)}} + \underbrace{(\tau_2^{0'} - \tau_1^2)}_{\propto m_2, p^{(2)}} + \underbrace{(\tau_3^1 - \tau_2^{0'})}_{\propto m_\emptyset, p^{(1)}} + \underbrace{(\tau_4' - \tau_3^1)}_{\propto m_1, p^{(1)}} \end{aligned}$$

for any limited input u, as both trajectories correspond to picking up and dropping off the objects individually in an arbitrary order. Once o_2 is detected, the actual optimal trajectory can be computed and, thus, $\tau_4 \geq \tau_3^{20}$ holds for any limited input u. Hence, an optimal solution is obtained by assuming $t^{(1)} \leq t^{(2)} \leq \tau_1$ in this case. Iteratively extending the set of objects O' by one object until $O' = O$ and following the same line of argumentation leads to (15) for the general case with L objects. □

As a consequence of Theorem 2, the computation of the exploration part and the pick-up and drop-off part of the control can be decoupled, similar to the single object case. The receding horizon solution of Problem 1 reduces to recomputing the control only once, after all objects have been detected. As for some applications it may be important to provide an estimate for the overall time to solve the complete pick-up and delivery task, e.g. for search and rescue operations, we also investigate how to compute cost estimates in the following.

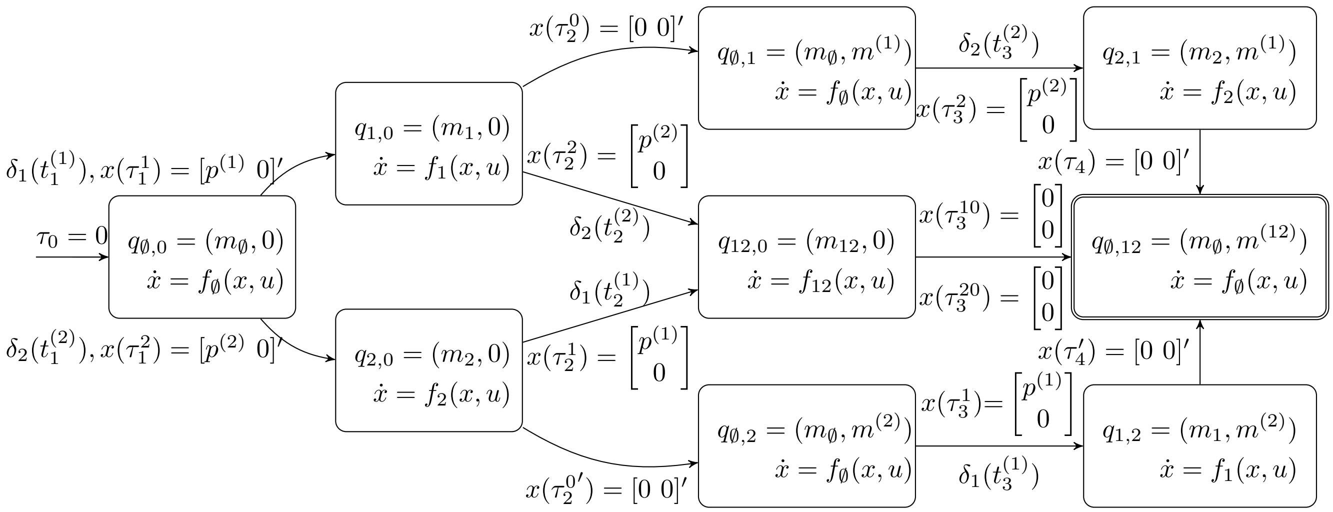


Fig. 1. Hybrid automaton for $O' = \{o_1, o_2\}$. Executions that solve the OCP's end in the highlighted final state.

A. Worst-case solution

The worst-case optimal control is initially $u_\emptyset^{(1)} = (1|_{[0,\sqrt{y_b m_\emptyset})}, -1|_{[\sqrt{y_b m_\emptyset}, t^{(l_1)})})$, where $t^{(l_1)}$ is the detection time of the last object, followed by $u_\emptyset^{(2)}$ and bang-bang controllers for picking up all objects with possible intermediate drop-offs at the depot. The following proposition holds.

Proposition 1. *Assuming that $O = \{o_1, \ldots, o_L\}$ is a totally ordered set such that*

$$m^{(l_1)} \leq m^{(l_2)}, p^{(l_1)} \leq p^{(l_2)}, \forall l_1, l_2 \in \{1, \ldots, L\}, \qquad (16)$$

yields the worst-case cost of an optimal execution $\xi|^*_{[0,T]}$.

Proof. Due to the monotonicity of the cost function (14) in both mass and distance, the maximal cost of an optimal execution that satisfies Theorem 2 is obtained when (16) holds for all objects. □

Intuitively, Proposition 1 relies on the worst-case assumption that objects with larger masses are located further away from the origin. The cost estimate is obtained by assuming costs similar to (7) and an optimization over all admissible discrete state sequences $\mathcal{Q}$, where the locations of the objects are free variables denoted by θ, i.e.

$$T_{\text{wc}} \leq \min \max_{\theta}\ T_{\mathcal{Q}}, \text{s.t. } \theta \in [y_a, y_b]^L \qquad (17)$$

that can be solved following the steps:

1) Find extremal candidates $\theta_j, \forall j, T_{\zeta_j}, \zeta_j \in \mathcal{Q}$ by applying first order optimality conditions with respect to θ;
2) If θ_j yields a local maximum of T_{ζ_j} and $T_{\zeta_j} \leq T_{\zeta_k}, \forall k, k \neq j$ at θ_j (global maximum), then θ_j is the certainty equivalent worst-case localization and ζ_j is the optimal discrete state sequence with duration T_{ζ_j}.

B. Probabilistic solution

From Theorem 2 it follows that the probabilistic optimal control is initially given by (12) until $t^{(l_1)}$ denoting the last detection time of an object $o_{l_1} \in O$, followed by a deterministic optimal control for pick-up and drop-off of all objects. A bound for the expected time is given as follows.

Proposition 2. *Assuming that $O = \{o_1, \ldots, o_L\}$ is a totally ordered set such that*

$$m^{(l_1)} \geq m^{(l_2)}, p^{(l_1)} \leq p^{(l_2)}, \forall l_1, l_2 \in \{1, \ldots, L\}, \qquad (18)$$

where $\forall o_{l_1}, o_{l_2} \in O$ yields a lower bound for the expected cost of an optimal execution $\xi|^*_{[0,T]}$.

Proof. Exploiting the monotonicity of the cost in both mass and distance, the lower bound for the expected cost of an optimal execution $\xi|^*_{[0,T]}$ that satisfies Theorem 2 is given when (18) holds for all objects. □

The minimum expected value can be computed analytically by solving multi-dimensional integrals over the object's interval, i.e.

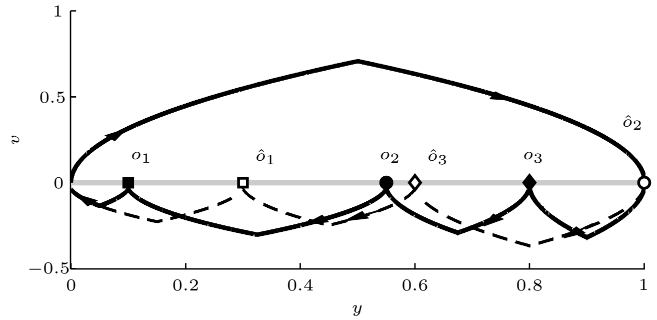
$$E\{T\} \geq \int_{p^{(L)}}^{y_b} \left(\ldots \left(\int_{y_a}^{p^{(1)}} \ell_2(q,x) dp^{(1)} \right) \ldots \right) dp^{(L)},$$

where $\ell_2(q,x)$ is a sum of costs corresponding to time-optimal bang-bang controllers over stages (e.g. as given in (7)), necessary to solve the overall OCP.

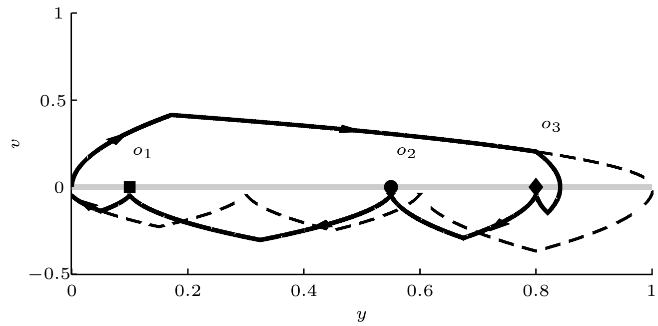
C. Numerical example

We now employ the proposed worst-case and probabilistic solutions in a numerical example. Let $O = \{o_1, o_2, o_3\}$ with $o_1 = (p^{(1)}, m^{(1)} = 1\,\text{kg})$, $o_2 = (p^{(2)}, m^{(2)} = 2\,\text{kg})$, $o_3 = (p^{(3)}, m^{(3)} = 1\,\text{kg})$ with $p^{(l)} \in [0,1]\,\text{m}, l \in \{1,2,3\}$. The actual locations of the objects are $p^{(1)} = 0.1\,\text{m}, p^{(2)} = 0.55\,\text{m}$ and $p^{(3)} = 0.8\,\text{m}$. The robot starts at Init$= ((m_\emptyset, 0)', [0\ 0]')$ with nominal mass $m_\emptyset = 2\,\text{kg}$ and moves according to the vector field set $F = \{f_\emptyset, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}, f_{123}\}$.

As we can see in Figure 2(a), the initial worst-case control guides the robot to erroneous estimated object positions $\hat{p}^{(1)} = 0.3$, $\hat{p}^{(2)} = 1$, $\hat{p}^{(3)} = 0.6$, while the actual solution (solid line) can be obtained once the real positions of all objects are known. As object detections may presumably



(a) Worst-case solution with cost $J^* = 9.6\,s$ (solid). The estimated worst-case object positions $\hat{p}^{(1)}$, $\hat{p}^{(2)}$ and $\hat{p}^{(3)}$ are denoted by □, ○ and ◇, respectively, and the predicted trajectory with cost $\hat{J}^* = 9.9s$ by the dashed line.



(b) Probabilistic solution with cost $J^* = 9.2\,s$ (solid). The dashed line represents one possible predicted trajectory, while the overall expected cost is $\hat{J}^* = 8.8\,s$.

Fig. 2. Comparison of the methods for three objects. The gray horizontal line denotes the object interval, the actual object positions are shown by a ■, ● and ◆ for $p^{(1)}$, $p^{(2)}$ and $p^{(3)}$, respectively.

occur at any point of the interval in the probabilistic case, the predicted trajectory (dashed) of the robot corresponds to covering the whole interval as shown in Figure 2(b), followed by an exemplary trajectory continuation driving all objects back to the depot. Once all objects are detected, the deterministic trajectory continuation solving the task can be computed (solid line), which is based on picking up o_3, then o_2 and then o_1, followed by a single drop-off at the depot in both cases. Note that the probabilistic control yields a better result than the worst-case control for this setup.

V. Discussion and Conclusions

A time-optimal hybrid control problem for a robot that has to find and collect a finite number of point-mass objects located on a line and move them to a depot has been addressed. A worst- and a probabilistic case, assuming uniform distribution of the objects over an interval, have been investigated. Necessary and sufficient optimality conditions have been derived for the single object case. Then, by exploiting properties of the problem at hand, it was shown that an optimal solution for several objects is based on complete exploration followed by a deterministic pick-up and drop-off of all objects, thus decoupling the exploration and the exploitation part of the control and reducing the computational effort significantly. Methods for computing cost estimates have also been proposed.

A simulation comparing both solutions has been provided, confirming the intuitive expectation that the control for the probabilistic case should yield a better result for a larger area of the interval. However, further simulations with a varying number of objects randomly located over intervals with different sizes have shown that this "hedging policy" only pays off for a small number of objects, while for cases when the density of the objects is relatively high compared to the size of the interval, the worst-case control will lead to a better solution. This suggests the existence of a setup dependent threshold for the performance of the worst-case and the probabilistic method, which will be a matter of further investigations. Future work will also address a generalization of the results for higher dimensions or other probability distributions.

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