Max-Consensus in a Max-Plus Algebraic Setting: The Case of Fixed Communication Topologies

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***Abstract*—Consensus algorithms have been studied in the field of distributed computing for decades. Recently consensus algorithms have attracted renewed attention because they can be exploited for distributed cooperative control. The purpose of this paper is the analysis of a specific class of consensus algorithms called max-consensus. This class of algorithms is needed for applications such as minimum time rendezvous and leader election. A new approach using max-plus algebra is proposed to analyze convergence of max-consensus algorithm. In this paper we focus on the problem of achieving max-consensus in time-invariant communication topologies. Conditions to achieve max-consensus are discussed and the convergence rate of the algorithm for different communication topologies is studied.**

I. Introduction

Various applications of distributed cooperative systems have been reported in the literature (e.g., [1], [2] and [3]). These include formation control ([4], [5] and [6]), rendezvous ([7], [8] and [9]), flocking ([10], [11] and [12]), cooperative search etc.

Cooperation in a network occurs if its elements share information over the (possibly time varying) network topology, and develop a consistent view regarding objectives and relevant information on the environment. To realize this, two ingredients are necessary. The *information state* is a network member's local estimate of the relevant variables, and the *consensus algorithm* describes how these local estimates change through information exchange between network members. A consensus is said to be reached if all local information states coincide.

Consensus algorithms, including average consensus, have been studied by numerous authors, e.g. [13], [14] and [15].

In some applications, such as minimum-time rendezvous or leader election, a special class of consensus algorithms called *max-consensus* has to be used. In this case a consensus is said to be achieved, if all the members in the network have the same information state, which is required to be the maximum of all initial information states.

An approach to analyze max-consensus is given in [15], where a nonlinear Log-Sum-Exp (LSE-) function is used to characterize min- or max-consensus as the limit behaviour of this LSE-function. Statements about the existence of convergence are given indirectly using a transformation to an averaging iterative process. In this contribution, we propose an approach to analyze max-consensus algorithms which is based on max-plus algebra (e.g., [16] and [17]). In this framework max-consensus algorithms become linear and may be analyzed easily.

The paper is organized as follows: Section II summarizes the necessary elements from graph theory and max-plus algebra. Section III introduces the basic concepts of max-consensus in a max-plus algebraic setting. Section IV provides convergence conditions for so-called strong max-consensus and investigates the convergence rate. Section V does the same for so-called weak max-consensus. Finally some simulation results and a conclusion are given in Section VI.

II. Graph Theory and Max-Plus Algebra

In this section we collect a few basic facts on graphs and max-plus algebra that are needed in the sequel.

A. Graph Theory

Information exchange between nodes in a network is modeled by means of *directed* or *undirected graphs*. A directed graph $\mathcal{G}$ is a pair $(\mathcal{N},\mathcal{E})$, where $\mathcal{N}=\{1,\cdots,n\}$ is a finite nonempty node set and $\mathcal{E}\subseteq\mathcal{N}\times\mathcal{N}$ is a set of ordered pairs of nodes, called edges. An edge may connect a node to itself (self loop). Existence of an edge $(j,i)\in\mathcal{E}$ denotes that node i can obtain information from node j. The edges $\mathcal{E}$ in an undirected graph have no orientation and are thus non-ordered pairs of nodes. Note that an undirected graph can be viewed as a special case of a directed graph, where an edge (j,i) in an undirected graph corresponds to edges (j,i) and (i,j) in the directed graph. If (j,i) is an edge in a directed graph, node j is called a *predecessor* of node i. $\mathcal{J}_i$ is the set of predecessor nodes of node i, i.e., $\mathcal{J}_i=\{j\in\mathcal{N}|(j,i)\in\mathcal{E}\}$. A *path* is a sequence of nodes $(i_1,\cdots,i_p)$, $p>1$, such that i_j is a predecessor of i_{j+1}, $j=1,\cdots,p-1$. An *elementary path* is a path in which no node appears more than once. A directed graph is said to be *strongly connected* if there is a path from any node to any other node in the graph, and it is called *weakly connected* if the graph obtained by adding an edge (i,j) for every existing edge (j,i) in the original graph is strongly connected. A *rooted directed tree* is a directed graph in which every node has exactly one predecessor node, called parent, except for one node, called the root, which has no parent and from which there is a directed path to every other node. $(\mathcal{N}_1,\mathcal{E}_1)$ is a *subgraph* of $(\mathcal{N},\mathcal{E})$, if $\mathcal{N}_1\subseteq\mathcal{N}$ and $\mathcal{E}_1$ is

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a set of edges (j,i) from $\mathcal{E}$, such that $i, j \in \mathcal{N}_1$. More details on graph theory can be found in, e.g., [18], [16].

B. Max-Plus Algebra

Max-plus algebra, e.g., [16] and [17], represents a powerful tool for simulation and analysis of timed cyclic discrete-event systems and allows for a compact representation of weighted graphs.

Max-plus algebra consists of two binary operations, $\oplus$ and $\otimes$, on the set $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$. The operations are defined as follows:

$$a \oplus b \quad := \quad max(a, b), \tag{1}$$
$$a \otimes b \quad := \quad a + b. \tag{2}$$

The neutral element of max-plus addition $\oplus$ is $-\infty$, denoted as ε. The neutral element of multiplication $\otimes$ is 0, denoted as e. The elements ε and e are also referred to as the zero and one element of max-plus algebra. Similar to conventional algebra, associativity, commutativity, and distributivity of multiplication over addition also hold for the max-plus algebra. Both operations can be extended to matrices in a straightforward way. For $A, B \in \mathbb{R}_{\max}^{m \times n}$,

$$(A \oplus B)_{ij} := a_{ij} \oplus b_{ij}, \quad i = 1, \cdots, m,\ j = 1, \cdots, n\ .$$

For $A \in \mathbb{R}_{\max}^{m \times n},\ B \in \mathbb{R}_{\max}^{n \times q}$,

$$(A \otimes B)_{ij} := \bigoplus_{k=1}^{n} (a_{ik} \otimes b_{kj}) = \max_{k} (a_{ik} + b_{kj}), \quad i = 1, \cdots, m,\ j = 1, \cdots, q.$$

Multiplication of a matrix $A \in \mathbb{R}_{\max}^{m \times n}$ and a scalar $\alpha \in \mathbb{R}_{\max}$ is defined by

$$(\alpha \otimes A)_{ij} := \alpha \otimes a_{ij} = \alpha + a_{ij}, \quad i = 1, \cdots, m,\ j = 1, \cdots, n.$$

Note that, as in conventional algebra, the multiplication symbol $\otimes$ is often omitted.

In the sequel, we also need matrices of zero elements, denoted by N, and of one elements, denoted by E. The identity matrix I is a square matrix with

$$(I)_{ij} := \begin{cases} \mathrm{e} & \text{for } i = j \\ \varepsilon & \text{else.} \end{cases}$$

For any matrix $A \in \mathbb{R}_{\max}^{n \times n}$, its *precedence graph* $\mathcal{G}(A)$ is defined in the following way: it has n nodes, denoted by $1, \cdots, n$, and (j,i) is an edge if and only if $a_{ij} \neq \varepsilon$. In this case a_{ij} is the weight of edge (j,i). Then

- A path in $\mathcal{G}(A)$ is a sequence of $p > 1$ nodes, denoted by $\rho := i_1, \ldots, i_p$, such that $a_{i_{k+1} i_k} \neq \varepsilon,\ k = 1, \ldots, p-1$.
- The weight of a path ρ denoted by $|\rho|_w$ is defined as $|\rho|_w := \sum_{k=1}^{p-1} a_{i_{k+1} i_k}$ and its corresponding length as $|\rho|_l := p - 1$.
- The length of the shortest path from node i to node j is denoted by $|i,j|_{l,min}$.
- $(A^k)_{ij}$ represents the maximal weight of all paths of length k from node j to node i, where

$$A^k := \underbrace{A \otimes A \otimes \ldots \otimes A}_{(k-1)\text{-times multiplication}}, \quad k \geq 1$$

and $A^0 = I$.

III. Max-Consensus in Fixed Topologies

The following example is introduced to illustrate the concept of max-consensus. A group of agents is assigned to meet simultaneously at a particular place. Each agent needs a certain *minimum time* to get to the meeting place. This means that the only possibility to achieve an agreement is to take the largest value of all the possible meeting times. A possible centralized solution to this problem could be a simultaneous communication between *all* agents in the group, to arrange a time when the group will meet. For this purpose the time of this communication should be known to the group. Thus, the centralized approach does not help to solve the problem. A distributed approach to the problem would be for each agent to communicate, one at a time, with a subset of the group. An update of his current estimate of the meeting time may be performed by taking the maximum of this estimate and that of the contacted agents. The question then is under which conditions does this strategy guarantee that all group members' estimates regarding the meeting time will converge.

To illustrate this, suppose there are eight agents in the group, who can communicate with exactly one other agent at each communication instant. The communication topology is given by the directed graph $\mathcal{G}(A)$ shown in Fig. 1. Self loops indicate that each node has access to its previous estimate. Fig. 2 shows how the time estimate of each group member evolves over time. Clearly, in this example a consensus is reached, and it is the maximum of all initial estimates, i.e., the maximum of all the required minimum time durations.

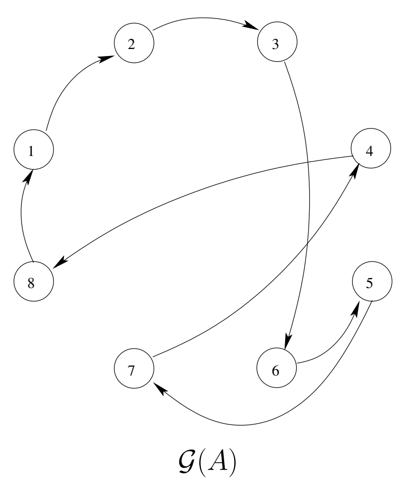

Fig. 1. Directed graph representing communication topology

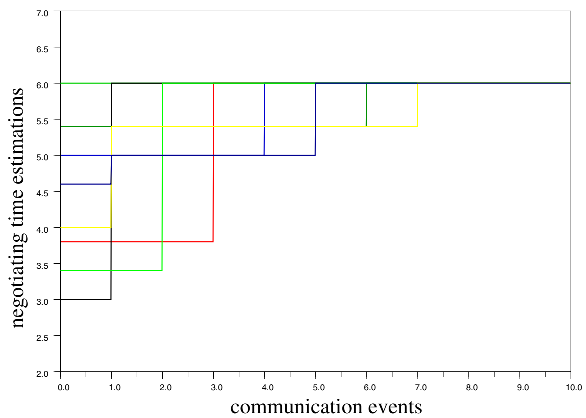

Fig. 2. Evolution of time estimates

Another application where a max-consensus algorithm can be useful is leader election. The network nodes communicate to decide which node should take the role of leader. For example, if each node has a performance index or can compute a performance index related to the task to be executed, the nodes can compare their performance indices. The node with the highest performance index is chosen to be the leader. This problem shares the same concepts and the same principles as the example above.

Basic Concepts

The two necessary ingredients for every consensus mechanism are the notion of *information state* and the actual *consensus algorithm*. The former represents locally available information, for example the current local estimate of the meeting time in rendezvous problems. The consensus algorithm dictates the way the information is processed and weighted by each network member.

The communication network between the members is modeled by a directed graph[1]. The existence of an edge (j, i) represents the ability of node i to receive information from node j. The weights of all edges are e, (i.e., 0 in conventional algebra). We assume that the communication topology is fixed. It is also assumed that communication occurs synchronously, i.e., each node exchanges information with its neighbors simultaneously.

We consider the discrete-time case, where communication instants may either be defined by a clock or by the occurrence of external events. The max-consensus algorithm is then simply:

$$x_i(k+1) = \max_{j \in \mathcal{J}_i} \{x_j(k)\}, \quad i = 1, \cdots, n, \tag{3}$$

where $\mathcal{J}_i$ is the set of predecessor nodes of node i and k denotes the kth communication event. We assume that $i \in \mathcal{J}_i$ $\forall i \in \mathcal{N}$, i.e., there exists a self loop for every node reflecting the fact that each node will include its own current estimate in the update process.

[1]All statements in this paper are based on this assumption and can easily be modified for the case of undirected graphs.

Definition 3.1 (Max-Consensus): Given a directed graph $\mathcal{G}$, an initial vector of information states $x(0) := (x_1(0), \cdots, x_n(0))^T$ and algorithm (3), *max-consensus* is said to be achieved, if $\exists l \in \mathbb{N}_0$ such that

$$\begin{aligned} x_i(k) &= x_j(k) \qquad (4) \\ &= \max\{x_1(0), \cdots, x_n(0)\}, \quad \forall k \geq l, \quad \forall i, j \in \mathcal{N}. \end{aligned}$$

If (4) holds for all $x(0)$, we say that *strong max-consensus* is achieved. If (4) only holds for a subset of all possible $x(0)$, *weak max-consensus* is achieved.

Clearly, strong max-consensus is a property of the graph $\mathcal{G}$, i.e., the network topology, only, whereas weak max-consensus depends on $\mathcal{G}$ and $x(0)$.

Eqn. (3) describes the temporal evolution of the information state. Clearly, (3) is nonlinear in conventional algebra. However, it becomes a linear equation in max-plus algebra:

$$x_i(k+1) = \bigoplus_{j \in \mathcal{J}_i} (x_j(k)), \quad i = 1, \cdots, n, \tag{5}$$

i.e.,

$$x(k+1) = A \otimes x(k), \tag{6}$$

where $x(k) = (x_1(k), \cdots, x_n(k))^T$ and A is a matrix in $\mathbb{R}_{\max}^{n \times n}$ such that $\mathcal{G}$ is its precedence graph. Recursive application of (6) provides

$$x(k) = A^k \otimes x(0).$$

IV. Strong max-consensus: Conditions and Properties

Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a matrix such that $\mathcal{G}(A)$ is the directed graph representing the communication topology. Recall that all edges in the graph have weight e and that there is a self loop in each node. Therefore all elements of A, hence of A^k, are either e or ε, and $a_{ii} = \mathrm{e}$, $i = 1, \cdots, n$. Clearly, there exists a path of length k from node j to node i if and only if $(A^k)_{ij} = \mathrm{e}$.

Theorem 4.1: A necessary and sufficient condition for strong max-consensus is that

$$A^l = E \quad \text{for some } l \in \mathbb{N}. \tag{7}$$

Proof : (7) implies $x(l) = E \otimes x(0)$, i.e.,

$$\begin{aligned} x_i(l) &= \bigoplus_{j=1,\cdots,n} (x_j(0)) \\ &= \max\{x_1(0), \cdots, x_n(0)\}, \quad i = 1, \cdots, n. \end{aligned}$$

Furthermore, applying the rules for multiplying matrices in max-plus algebra provides

$$A^{k+l} = E \quad \forall k \geq 0,$$

and therefore:

$$x_i(k+l) = \max\{x_1(0), \cdots, x_n(0)\}, \quad i = 1, \cdots, n.$$

This shows sufficiency. Necessity is obvious: if $A^l \neq E\ \forall l$, then, $(\forall l)\ \exists i_l, j_l \in \mathcal{N}$ s.t. $(A^l)_{i_l j_l} = \varepsilon$, i.e., $x_{i_l}(l)$ does not depend on $x_{j_l}(0)$. Hence, if $x_{j_l}(0)$ is the maximum element of $\{x_1(0), \cdots, x_n(0)\}$, (4) will not hold. □

Note that the condition in Theorem 4.1 is equivalent to the requirement that $\mathcal{G}(A)$ is strongly connected: clearly, $A^l = E$ implies that there exists a path of length l between any pair of nodes in $\mathcal{G}(A)$, i.e., the graph is strongly connected. On the other hand, if $\mathcal{G}(A)$ is strongly connected, there exists an elementary path between any two nodes. Take l to be the maximum of the length of all elementary paths. Then, because of self loops on each node, we can have paths of length l for any pair of nodes, i.e., $A^l = E$.

Corollary 4.1: Let $A \in \mathbb{R}_{\max}^{n\times n}$, with $a_{ij} \in \{\varepsilon, \mathrm{e}\}$ and $a_{ii} = \mathrm{e}$. If there exists an l, such that $A^l = E$ and $A^k \neq E$ for all $k < l$, then

$$l \leq n - 1\ . \tag{8}$$

Proof : Recall that $(A^l)_{ij} = \mathrm{e}$ if and only if there exists a path in $\mathcal{G}(A)$ from node j to node i with length l, or equivalently, there exists an elementary path from node j to node i with length less or equal to l. As the maximum length of an elementary path in $\mathcal{G}(A)$ is $n-1$, (8) holds. □

Example 4.1: Consider the two strongly connected directed graphs shown in Fig. 3. The associated matrices are as follows:

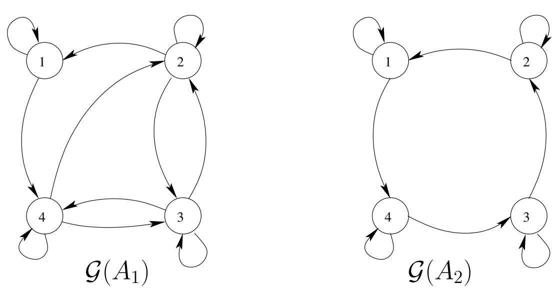

Fig. 3. Example 4.1

$$A_1 = \begin{pmatrix} \mathrm{e} & \mathrm{e} & \varepsilon & \varepsilon \\ \varepsilon & \mathrm{e} & \mathrm{e} & \mathrm{e} \\ \varepsilon & \mathrm{e} & \mathrm{e} & \mathrm{e} \\ \mathrm{e} & \varepsilon & \mathrm{e} & \mathrm{e} \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} \mathrm{e} & \mathrm{e} & \varepsilon & \varepsilon \\ \varepsilon & \mathrm{e} & \mathrm{e} & \varepsilon \\ \varepsilon & \varepsilon & \mathrm{e} & \mathrm{e} \\ \mathrm{e} & \varepsilon & \varepsilon & \mathrm{e} \end{pmatrix}.$$

For these matrices one can easily check that $A_1^k = E$ for $k \geq 2$ and $A_2^k = E$ for $k \geq 3$. As expected from Corallary 4.1 there exists an $l \leq n - 1 = 3$ for both graphs such that $A^l = E$.

The smallest possible l meeting the condition $A^l = E$ denotes the necessary and sufficient number of communication instants to reach strong max-consensus. $\frac{1}{l}$ is therefore referred to as the *convergence rate* of the max-consensus algorithm. l can be characterized as follows:

Corollary 4.2: Let $A \in \mathbb{R}_{\max}^{n\times n}$, with $a_{ij} \in \{\varepsilon, \mathrm{e}\}$ and $a_{ii} = \mathrm{e}$. If there exists an l such that $A^l = E$ and $A^k \neq E$ for all $k < l$, then

$$l = \max_{i,j=1,\ldots,n} \{|i,j|_{l,min}\}, \tag{9}$$

where $|i,j|_{l,min}$ denotes the length of the shortest existing path from node i to node j.

Proof : Pick any pair i, j such that $(A^{l-1})_{ji} = \varepsilon$. As, by assumption, $(A^l)_{ji} = e$, l is the length of the shortest path from node i to node j in $\mathcal{G}(A)$. Because of $A^l = E$, there exists a path of length l or shorter between any other pair of nodes. Hence (9) holds. □

Example 4.2: Note that in $\mathcal{G}(A_1)$ in Example 4.1, the shortest path length between any two nodes is 2, while the shortest path length between any two nodes of $\mathcal{G}(A_2)$ is 3.

In summary, we have shown that *strong max-consensus* is achieved if and only if $\mathcal{G}(A)$ is strongly connected and that the required number of communication instants is the maximum of the shortest path length between any pair of nodes in $\mathcal{G}(A)$.

V. Weak max-consensus: Conditions and Properties

For weak max-consensus, strong connectivity of the graph is sufficient but not necessary:

Theorem 5.1: Let $A \in \mathbb{R}_{\max}^{n\times n}$, with $a_{ij} \in \{\varepsilon, \mathrm{e}\}$ and $a_{ii} = \mathrm{e}$. Denote the maximum of all initial states by $\hat{x}_0$, i.e.,

$$\hat{x}_0 := \max_{i=1,\cdots,n} \{x_i(0)\}$$

and partition the index set $\mathcal{N} = \{1, \cdots, n\}$ by:

$$\begin{aligned} \mathcal{N}_1 &:= \{i | x_i(0) = \hat{x}_0\}, \\ \mathcal{N}_2 &:= \mathcal{N} \setminus \mathcal{N}_1. \end{aligned}$$

Then, a necessary and sufficient condition for max-consensus is:

$$(\exists l \in \mathbb{N}) \quad (\forall i \in \mathcal{N}_2) \quad (\exists m \in \mathcal{N}_1) : (A^l)_{im} = \mathrm{e}\ . \tag{10}$$

Proof : To show sufficiency, first note that (10) implies

$$(A^{l+k})_{im} = \mathrm{e}\ , k \in \mathbb{N}.$$

Furthermore, we have $\forall i \in \mathcal{N}_2$

$$\begin{aligned} x_i(l+k) = & \bigoplus_{j\in\mathcal{N}_1} (A^{l+k})_{ij} \otimes \underbrace{x_j(0)}_{\hat{x}_0} \\ & \bigoplus_{j\in\mathcal{N}_2} (A^{l+k})_{ij} \otimes x_j(0) \\ = & \qquad \hat{x}_0. \end{aligned}$$

As $(A^{l+k})_{ii} = \mathrm{e}$, we also have

$$x_i(l+k) = \hat{x}_0 \quad \forall i \in \mathcal{N}_1.$$

To show necessity, assume that (10) does not hold, i.e.,

$$(\forall l \in \mathbb{N}) \quad (\exists i \in \mathcal{N}_2) \quad (\forall m \in \mathcal{N}_1) : (A^l)_{im} = \varepsilon.$$

This implies that there exists a node i in $\mathcal{N}_2$ such that, $\forall l$, $x_i(l)$ does not depend on $\hat{x}_0$, hence consensus is not achieved. □

Note that the condition in Theorem 5.1 is equivalent to the requirement that there exist subgraphs $\mathcal{G}_{T_i} := (\mathcal{N}_{T_i}, \mathcal{E}_{T_i})$, of graph $\mathcal{G}(A)$ such that the G_{T_i} are rooted directed trees with roots in $\mathcal{N}_1$ and $\cup_i \mathcal{N}_{T_i} \supseteq \mathcal{N}_2$: (10) is equivalent to the fact that for any node in $\mathcal{N}_2$, there exists a path of length l ending in this node and starting in a node in $\mathcal{N}_1$, i.e., there exist directed trees with the above properties.

Corollary 5.1: Let $A \in \mathbb{R}_{\max}^{n\times n}$, with $a_{ij} \in \{\varepsilon, \mathrm{e}\}$ and $a_{ii} = \mathrm{e}$. Let l_{min} be the smallest l such that (10) holds. Then

$$l_{min} \leq n-1. \tag{11}$$

Proof : Recall that $(A^{l_{min}})_{im} = \mathrm{e}$ if and only if there exists a path in $\mathcal{G}(A)$ from node m to node i with length l_{min} or, equivalently, there exists an elementary path from node m to node i with length less or equal to l_{min}. As the maximal length of an elementary path in $\mathcal{G}(A)$ is $n-1$, (11) holds. □

Example 5.1: Consider the weakly connected directed graph in Fig. 4. Assume that $x_1(0) = x_2(0) = \hat{x}_0$ and $x_i(0) < \hat{x}_0$, $i = 3, \cdots, 6$. Hence $\mathcal{N}_1 = \{1, 2\}$ and $\mathcal{N}_2 = \{3, 4, 5, 6\}$. Clearly, there exist two rooted directed trees with nodes 1 and 2 as roots such that $\mathcal{N}_2$ is contained in the union of the node sets of the trees. We therefore have max-consensus. This is also reflected in the corresponding A-matrix:

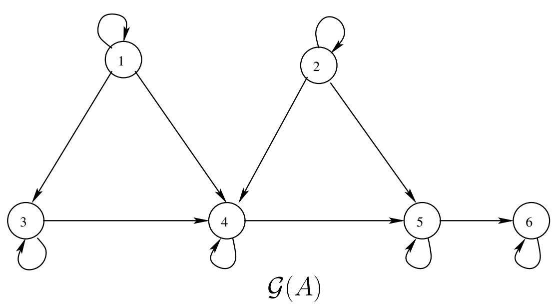

Fig. 4. Example 5.1

$$A = \begin{pmatrix} \mathrm{e} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \mathrm{e} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \mathrm{e} & \varepsilon & \mathrm{e} & \varepsilon & \varepsilon & \varepsilon \\ \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \varepsilon & \varepsilon \\ \varepsilon & \mathrm{e} & \varepsilon & \mathrm{e} & \mathrm{e} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \mathrm{e} & \mathrm{e} \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \mathrm{e} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \mathrm{e} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \mathrm{e} & \varepsilon & \mathrm{e} & \varepsilon & \varepsilon & \varepsilon \\ \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \varepsilon & \varepsilon \\ \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \varepsilon \\ \varepsilon & \mathrm{e} & \varepsilon & \mathrm{e} & \mathrm{e} & \mathrm{e} \end{pmatrix},$$

i.e., condition (10) is satisfied.

We now precisely characterize the smallest required number of communication events to achieve weak max-consensus.

Corollary 5.2: Let $A \in \mathbb{R}_{\max}^{n\times n}$, with $a_{ij} \in \{\varepsilon, \mathrm{e}\}$ and $a_{ii} = \mathrm{e}$. Let l_{min} be the smallest l such that (10) holds. Then

$$l_{min} = \max_{i\in\mathcal{N}_2}(\min_{m\in\mathcal{N}_1}\{|m,i|_{l,min}\}) \tag{12}$$

Proof : (12) implies that for any $i \in \mathcal{N}_2$ there exists a path of length l_{min} starting in $\mathcal{N}_1$ and ending in i. Therefore $\exists m \in \mathcal{N}_1$ s.t.

$$(A^{l_{min}})_{im} = \mathrm{e}.$$

If l is less than l_{min}, there is at least one $i \in \mathcal{N}_2$ s.t. there exists no path of length l from $\mathcal{N}_1$ to i. Hence (10) does not hold. □

VI. Simulation Results and Conclusions

A. Simulation Results

1) Strong Max-Consensus:: Consider the graph in Fig. 5, where for simplicity, we have omitted the self loops. The graph is strongly connected with

$$l_{min} = \max_{i,j\in\mathcal{N}} |i,j|_{l,min} = 20.$$

Hence max-consensus will be achieved for any initial state after 20 communication instants. Fig. 6 and Fig. 7 show the simulation results for two different initial information states. In Fig. 6 the algorithm converges to the maximum value of all initial information states in 20 communication instants, which illustrates the worst-case, whereas it converges in 12 communication instants for the initial information state illustrated in Fig. 7.

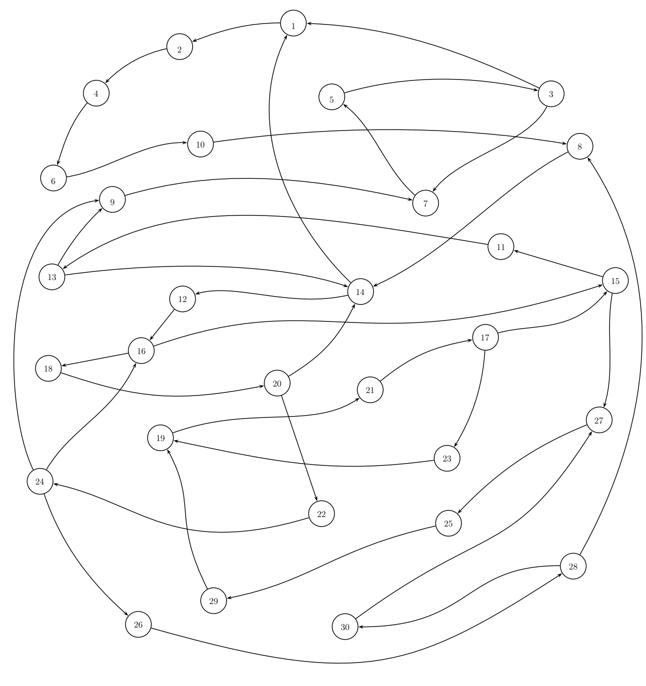

Fig. 5. A strongly connected graph with 30 nodes. The self loops in each node are not shown.

2) Weak Max-Consensus:: The left part of Fig. 8 shows a weakly connected directed graph. According to the results from Section V, max-consensus will be achieved if there exist subgraphs $\mathcal{G}_{T_i} = (\mathcal{N}_{T_i}, \mathcal{E}_{T_i})$ of $\mathcal{G}$ such that the roots of $\mathcal{G}_{T_i}$ are in $\mathcal{N}_1$ and $\cup_i \mathcal{N}_{T_i} \supseteq \mathcal{N}_2$. In this example, this condition is met if $x_5(0) > x_i(0)$, $i = 1, \cdots, 4$, and the corresponding tree is shown in Fig. 8. Simulation results are given in Fig. 9.

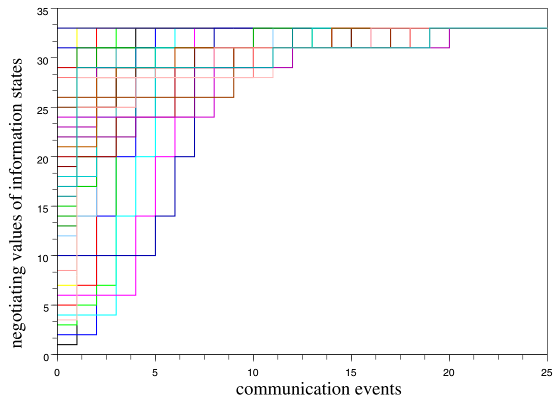

Fig. 6. Simulation results for a network topology with 30 agents

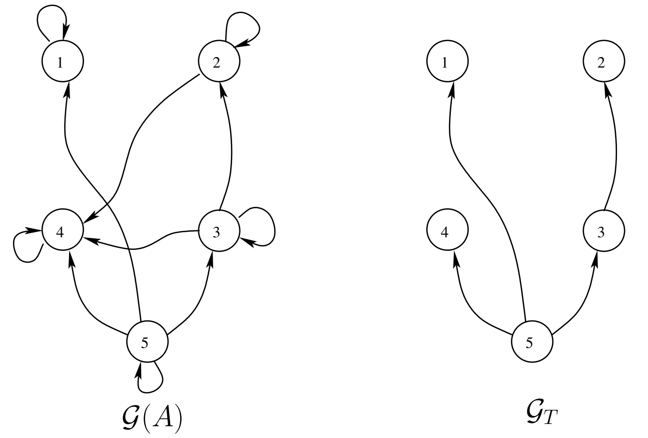

Fig. 8. A weakly connected graph $\mathcal{G}(A)$ and a subgraph $\mathcal{G}_T$ that is a rooted directed tree

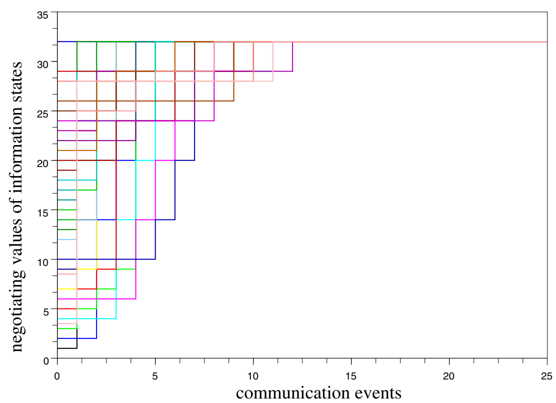

Fig. 7. Simulation results for a network topology with 30 agents

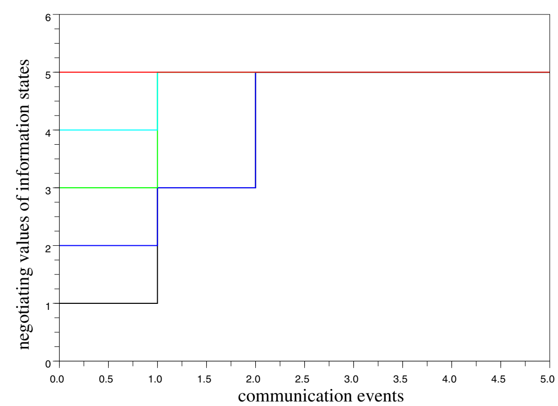

Fig. 9. Simulation results for a network topology given in Fig. 8

B. Conclusions

A main ingredient in any distributed cooperative control system is an efficient consensus mechanism. In this paper, we have investigated max-consensus problems, where one aims at determining the maximal value of all agents' initial information states through local communication.

We have proposed to use max-plus algebra to analyze max-consensus algorithms. In this framework, we obtain a linear system representation of the investigated consensus algorithm. We have studied convergence conditions and convergence rates for both strong and weak max-consensus for time-invariant (fixed) communication topologies. Note that this approach can also be used to study min-consensus problems – one then seeks the maximum of $-x_i(0)$, $i = 1, \cdots, n$.

An extension to time-variant (switching) communication topologies is reported elsewhere ([19]).

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