Pole-Placement Design – A State-Space Approach

Overview

• Control-System Design
• Regulation by State Feedback
• Observers
• Output Feedback
• The Servo Problem
Control System Design

- **Regulation problem**: The major issue in the regulation problem is to compromise between reduction of load disturbances and the fluctuations created by the measurement noise that is injected in the system due to feedback.

- **Servo problem**: The command signal following is the major issue in servo problems.

- In this lecture we will develop a design method based on state-space models whose purpose is to obtain a specific closed-loop characteristic polynomial of the system.

### The Process

Let us start with a simple design problem first. The SISO plant is given by

\[
\frac{dx}{dt} = Ax + Bu
\]

Sampling yields the discrete-time system

\[
x[k + 1] = \Phi x[k] + \Gamma u[k]
\] (1)
with

\[ \Phi = e^{A\Delta}, \quad \Gamma = \int_0^\Delta e^{As} ds B \]

**Disturbances**

- Initial assumption: Impulses that are widely spread so that system can settle between impulses and effect can be represented as initial state

**Criterion**

- Bring the state to zero after perturbations in the initial condition
- The rate of decay of the state is given indirectly by specifying the poles of the closed-loop system
- Servo problems: reference model for signal transmission from command signals to process variables
Admissible controls

• First we consider static state-feedback

\[ u = -Lx[k] \]

• Later we extend this to dynamic state feedback and output feedback.

Design parameters

• Sampling period
• Closed-loop poles
• Trade-off between the magnitude of control signals and the speed at which the systems recovers from a disturbance.
Regulation by State Feedback

- Assumptions: Discrete-time model and disturbances introduced before
- See example ...
- Characteristic polynomial of the matrix $\Phi$:

$$z^n + a_1 z^{n-1} + \cdots + a_n$$

Assume the system is reachable so it can be transformed to reachable canonical form through the transformation $z = Tx$:

$$z[k + 1] = \tilde{\Phi} z[k] + \tilde{\Gamma} u[k]$$

(2)
where

\[
\tilde{\Phi} = \begin{pmatrix}
-a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}, \quad \tilde{\Gamma} = \begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix}
\]

The feedback law

\[
u = -\tilde{L}z[k] = -\left( p_1 - a_1 \quad p_2 - a_2 \quad \cdots \quad p_n - a_n \right) z[k]
\]

gives the characteristic closed-loop polynomial

\[
P(z) = z^n + p_1 z^{n-1} + \cdots + p_n.
\]

The solution for the original problem:

\[
u = -\tilde{L}z = -\tilde{L}Tx = -Lx
\]
Using

$$\mathbf{W}_c = \begin{pmatrix} \Gamma & \Phi \Gamma & \cdots & \Phi^{n-1} \Gamma \end{pmatrix}$$

and

$$\mathbf{T} = \tilde{\mathbf{W}}^{-1}_c \mathbf{W}_c$$

with

$$\mathbf{\tilde{W}}^{-1}_c = \begin{pmatrix} 1 & a_1 & \cdots & a_{n-1} \\ 0 & 1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

helps to determine $\mathbf{T}$. $\mathbf{\tilde{W}}_c$ is the reachability matrix of the system in $z$-coordinates. The control law can be rewritten as

$$\mathbf{L} = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} \mathbf{\tilde{W}}^{-1}_c \mathbf{W}_c$$
Theorem - Pole-Placement using State Feedback: Consider the system (1). Assume that there is only one input signal. If the system is reachable there exits a linear feedback that gives a closed-loop system with the characteristic polynomial $P(z)$. The feedback is given by

$$u[k] = -Lx[k]$$

with

$$L = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} \tilde{W}_c^{-1} W_c$$

$$= \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} W_c^{-1} P(\Phi) \quad (3)$$

where $W_c$ and $\tilde{W}_c$ are the reachability matrices of the systems (1) and (2), respectively.

Equation (3) is called Ackermann’s formula.

See example ...
Deadbeat Control

If the desired poles are all chosen to be at the origin, the characteristic polynomial of the closed-loop system becomes

\[ P(z) = z^n. \]

The system matrix of the closed-loop system satisfies

\[ \Phi^n_c = \Phi - \Gamma L = 0 \]

- This strategy has the property that it will drive all states to zero in at most \( n \) steps after an impulse disturbance in the system state.
- The control strategy is called deadbeat control.
- There is only one design parameter – the sampling period.
- The settling time is given by \( n \cdot \Delta \).
Taking into account disturbances

- In order to handle more general disturbances than impulse we assume

\[
\frac{dx}{dt} = Ax + Bu + v
\]

where \( v \) is a disturbance described by

\[
\frac{dw}{dt} = A_w w
\]
\[
v = C_w w
\]

with given initial conditions.

- \( A_w = 0 \): disturbance \( v \) is a constant.

- \( A_w = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \): sinusoidal disturbance

- It is assumed that \( w \) can be measured. This assumption is relaxed later.
Augmented state vector:

\[ z = \begin{pmatrix} x \\ w \end{pmatrix} \]

Complete system:

\[
\frac{d}{dt} \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} A & C_ω \\ 0 & A_w \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u
\]

After sampling:

\[
\begin{pmatrix} x[k+1] \\ w[k+1] \end{pmatrix} = \begin{pmatrix} \Phi & \Phi x_w \\ 0 & \Phi_w \end{pmatrix} \begin{pmatrix} x[k] \\ w[k] \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u[k]
\]

General linear state feedback:

\[
u[k] = -Lx[k] - L_w w[k]
\]
Resulting closed-loop system:

\[ x[k + 1] = (\Phi - \Gamma L) x[k] + (\Phi x_w - \Gamma L_w) w[k] \]
\[ w[k + 1] = \Phi_w w[k] \]

- Combination of a feedback term \( Lx \) and a feedforward term \( L_w w \)
- If the pair \((\Phi, \Gamma)\) is reachable, the matrix \( L \) can be chosen so that the matrix \((\Phi - \Gamma L)\) has prescribed eigenvalues.
- The matrix \( \Phi_w \) can not be influenced by feedback.
- By proper choice of \( L_w \), the influence of disturbances on the state vector \( x \) can be made small.
- Select \( L_w \) that makes \((\Phi x_w - \Gamma L_w)\) small or even zero.

See example ...
• Calculate or reconstruct the state $x[k]$ from input and output sequences $y[k], y[k - 1], \ldots, u[k], u[k - 1], \ldots$

• This is possible if the system

$$x[k + 1] = \Phi x[k] + \Gamma u[k]$$

$$y[k] = C x[k]$$

is observable.
Direct Calculation of the State Variables

\[ y[k - n + 1] = Cx[k - n + 1] \]
\[ y[k - n + 2] = C\Phi x[k - n + 1] + C\Gamma u[k - n + 1] \]
\[ \vdots \]
\[ y[k] = C\Phi^{n-1} x[k - n + 1] + C\Phi^{n-2} \Gamma u[k - n + 1] + \cdots + C\Gamma u[k - 1] \]

With

\[ Y_k = \begin{pmatrix} y[k - n + 1] \\ y[k - n + 2] \\ \vdots \\ y[k] \end{pmatrix}, \quad U_{k-1} = \begin{pmatrix} u[k - n + 1] \\ u[k - n + 2] \\ \vdots \\ u[k - 1] \end{pmatrix} \]
and

\[
W_0 = \begin{pmatrix}
C \\
C\Phi \\
C\Phi^2 \\
: \\
C\Phi^{n-1}
\end{pmatrix}
\quad W_u = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
C\Gamma & 0 & \cdots & 0 \\
C\Phi\Gamma & C\Gamma & \cdots & 0 \\
: & : & \ddots & : \\
C\Phi^{n-2}\Gamma & C\Phi^{n-3}\Gamma & \cdots & C\Gamma
\end{pmatrix}
\]

we can write in vector form:

\[
x[k - n + 1] = W_o^{-1}Y_k - W_o^{-1}W_u U_{k-1}.
\]

The state has been obtained in terms of future inputs and measurements.

Repeated use of the system state equation yields:

\[
x[k] = \Phi^{n-1}x[k - n + 1] + \Phi^{n-2}\Gamma u[k - n - 1] + \cdots + \Gamma u[k - 1]
\]
This can be expressed as

\[ x[k] = A_y Y_k + B_u U_{k-1} \]

with

\[ A_y = \Phi^{n-1} W_o^{-1}, \quad B_u = \left( \Phi^{n-2} \Gamma \quad \Phi^{n-3} \Gamma \quad \cdots \quad \Gamma \right) - \Phi^{n-1} W_o^{-1} W_u \]

The state vector is a linear combination \( y[k], y[k - 1], \ldots y[k - n + 1] \) and \( u[k - 1], \ldots u[k - n + 1] \).
Reconstruction Using a Dynamic System

- Direct calculation is sensitive to disturbances (due to differences)

- State \( x \) is to be approximated by the state \( \hat{x} \) of the model

\[
\hat{x}[k + 1] = \Phi \hat{x}[k] + \Gamma u[k]
\]  

(6)

- Good idea as long model is perfect and initial state is the same.

- Convergence from wrong initial state to true state only for asymptotically stable system

- Improving the state reconstruction by using the measured output

\[
\hat{x}[k + 1|k] = \Phi \hat{x}[k] + \Gamma u[k] + K (y[k] - C \hat{x}[k|k - 1])
\]  

(7)

where \( \hat{x}[k|k - 1] \) is the predicted state for time instant \( k \) based on available information at time instant \( k - 1 \)
To determine $K$ we look at the reconstruction error

$$\tilde{x} = x - \hat{x}$$

(8)

and its dynamics

$$\tilde{x}[k+1|k] = \Phi \tilde{x}[k|k-1] - K (y[k] - C \hat{x}[k|k-1])$$

(9)

$$= (\Phi - KC) \tilde{x}[k|k-1]$$

(10)

If $K$ is chosen such that the system before is asymptotically stable, the error $\tilde{x}$ will always converge to zero.

Eq. (7) is called an observer for the system (5)

Problem: Find a matrix $K$ such that the matrix $(\Phi - KC)$ has prescribed eigenvalues

$(\Phi - KC)$ and $(\Phi^T - C^T K^T)$ have the same eigenvalues and the second formulation is similar to the design of a state feedback

Translating previous results, the problem can be solved if the following matrix has full rank $n$:

$$W_o^T = \begin{pmatrix} C^T & \Phi^T C^T & \ldots & (\Phi^{n-1})^T C^T \end{pmatrix}$$

(11)
Theorem - Observer Dynamics: Consider the discrete-time system given in Eq. (5). Let $P(z)$ be a polynomial of degree $n$, where $n$ is the order of the system. Assuming that the system is completely observable, then there exists a matrix $K$ such that the matrix $(\Phi - KC)$ of the observer has the characteristic polynomial $P(z)$.

Computation of the Observer Gain

\[
L \rightarrow K^T \quad W_c \rightarrow W_o^T \quad \Phi \rightarrow \Phi^T
\]

\[
K^T = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} \left( W_o^T \right)^{-1} P(\Phi^T)
\]

or

\[
K = P(\Phi) W_o^{-1} \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^T
\]
A Deadbeat Observer

• If the observer gain $K$ is chosen so that the matrix $(\Phi - KC)$ has all eigenvalues zero, the observer is called a deadbeat observer.

• The observer error will go to zero in finite time.

• A deadbeat observer is equivalent to the direct computation of the state.

An Observer without Delay

• The previous observer had a delay of one step, because $\hat{x}[k|k-1]$ depends only on measurements up to time $k-1$.

• Observer to avoid a delay:

$$
\hat{x}[k|k] = \Phi \hat{x}[k-1|k-1] + \Gamma u[k-1]
$$

$$
+ K [y[k] - C(\Phi \hat{x}[k-1|k-1] + \Gamma u[k-1])]
$$

$$
= (I - KC)(\Phi \hat{x}[k-1|k-1] + \Gamma u[k-1]) + Ky[k]
$$
• Observer error

\[ \tilde{x}[k|k] = x[k] - \hat{x}[k|k] = (\Phi - KC\Phi)\tilde{x}[k - 1|k - 1] \]  

(13)

• The pair \((\Phi, C\Phi)\) is observable if the pair \((\Phi, C)\) is observable! This implies that arbitrary eigenvalues can be given to \((\Phi, C\Phi)\) by selection \(K\).

\[ y(k) - C\hat{x}[k|k] = C\tilde{x}[k|k] = (I - CK)C\Phi\tilde{x}[k - 1|k - 1] \]  

(14)

• If the system has one output, then \((I - CK)\) is a scalar. \(K\) may be chosen such that \(CK = 1\). This implies that \(y(k) = C\hat{x}[k|k]\). This will make it possible to eliminate one equation from (12). Reduced-order observes of this type are called Luenberger observers.
Output Feedback

- Before it was assumed that all states are measurable.
- Now we combine the state feedback with the observer to deal with non-measurable but observable states.
- System:

\[
\begin{align*}
    x[k + 1] &= \Phi x[k] + \Gamma u[k] \\
    y[k] &= Cx[k]
\end{align*}
\]  

- Aim: linear feedback law relating \( u \) to \( y \) such that the closed-loop system has given poles
- Initial assumption: impulse-like disturbances or equivalently unknown initial state
• Intuitive control law, when the state is not measurable

\[ u[k] = -L \hat{x}[k|k-1] \]  

(16)

where \( \hat{x} \) is obtained from the observer

\[ \hat{x}[k+1|k] = \Phi x[k] + \Gamma u[k] + K (y[k] - C \hat{x}[k|k-1]) \]  

(17)

• Thus the feedback controller is a dynamic system of order \( n \).
**Analysis of the Closed-Loop System**

Using

\[
\tilde{x} = x - \hat{x}
\]  

(18)

and inserting the last control law into the considered system yields

\[
x[k + 1] = (\Phi - \Gamma L)x[k] + \Gamma L\tilde{x}[k|k - 1]
\]  

(19)

\[
\tilde{x}[k + 1|k] = (\Phi - K C)\tilde{x}[k|k - 1]
\]  

(20)

- The closed-loop system has the order \(2n\).
- The eigenvalues of the closed-loop system are the eigenvalues of the matrices \((\Phi - \Gamma L)\) and \((\Phi - K C)\).
- Design of state feedback and observer can be separated (separation principle).
- The controller can be expressed as \(n\)-th-order pulse-transfer function from \(y\) to \(u\):

\[
H_c(z) = -L(zI - \Phi + \Gamma L)^{-1}K
\]  

(21)
Output feedback without time-delay

- Use of the feedback law

\[ u[k] = -L \hat{x}[k|k] \]  \hspace{1cm} (22)

Together with the observer (12).

- The resulting closed-loop system has similar properties compared to the case before.
Extended system with disturbances

\[
\begin{pmatrix}
  x[k + 1] \\
  w[k + 1]
\end{pmatrix}
= \begin{pmatrix}
  \Phi & \Phi_{xw} \\
  0 & \Phi_w
\end{pmatrix}
\begin{pmatrix}
  x[k] \\
  w[k]
\end{pmatrix}
+ \begin{pmatrix}
  \Gamma \\
  0
\end{pmatrix} u[k]
\]

\[
y[k] = \begin{pmatrix}
  C \\
  0
\end{pmatrix}
\begin{pmatrix}
  x[k] \\
  w[k]
\end{pmatrix}
\]

Control law:

\[
u[k] = -L \hat{x}[k] - L_w \hat{w}[k]
\]

Observer:

\[
\begin{pmatrix}
  \hat{x}[k + 1] \\
  \hat{w}[k + 1]
\end{pmatrix}
= \begin{pmatrix}
  \Phi & \Phi_{xw} \\
  0 & \Phi_w
\end{pmatrix}
\begin{pmatrix}
  \hat{x}[k] \\
  \hat{w}[k]
\end{pmatrix}
+ \begin{pmatrix}
  \Gamma \\
  0
\end{pmatrix} u[k] + \begin{pmatrix}
  K \\
  K_w
\end{pmatrix} \varepsilon[k]
\]

\[
\varepsilon[k] = y[k] - C \hat{x}[k]
\]
Closed-loop system:

\[
\begin{align*}
x[k+1] &= (\Phi - \Gamma L)x[k] + (\Phi x - \Gamma L_w)w[k] + \Gamma L \ddot{x}[k] + \Gamma L_w \ddot{w}[k] \\
w[k+1] &= \Phi_w w[k] \\
\ddot{x}[k+1] &= (\Phi - K C) \ddot{x}[k] + \Phi x \ddot{w}[k] \\
\ddot{w}[k+1] &= \Phi_w \ddot{w}[k] - K_w C \ddot{x}[k]
\end{align*}
\]

- Disturbance state is observable but not reachable.
- \( L \) ensures decay of \( x \).
- \( L_w \) reduces the effect of disturbances (related to state \( w \)).
- \( K \) and \( K_w \) ensure that the observer error goes to zero.

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Integral Action

- Constant, but unknown, input disturbance \( v = w \leadsto \Phi x_w = \Gamma \) and \( \Phi w = 1 \).

- \( L_w = 1 \) gives perfect disturbance cancellation.

\[
\begin{align*}
  u[k] &= -L\hat{x}[k] - L_w\hat{w}[k] = -L\hat{x}[k] - \hat{w}[k] \\
  \hat{x}[k + 1] &= \Phi\hat{x}[k] + \Gamma(u[k] + \hat{w}[k]) + K\varepsilon[k] \\
  \hat{w}[k + 1] &= \hat{w}[k] + K_w\varepsilon[k] \\
  \varepsilon[k] &= y[k] - C\hat{x}[k]
\end{align*}
\]
The Servo Problem

- The objective is to make the states and outputs of the system respond to command signals in a specific way.

1st Approach

\[ u[k] = -L\hat{x}[k|k-1] + L_cu_c[k] \]  \hspace{1cm} (27)

- The observer is given by Eq. (7).

- Resulting closed-loop system:

\[ x[k+1] = (\Phi - GL)x[k] + GL\hat{x}[k] + GL_cu_c[k] \]  \hspace{1cm} (28)

\[ \tilde{x}[k+1] = (\Phi - KC)\tilde{x}[k] \]  \hspace{1cm} (29)

\[ y[k] = Cx[k] \]  \hspace{1cm} (30)

- Observer error is not reachable from \( u_c \). This is good.
Transfer function for the controlled system:

\[ H_{cl}(z) = C(zU - \Phi - \Gamma L)^{-1} \Gamma L = Lc \frac{B(z)}{A_{cl}(z)} \]

- \( Lc \) might be chosen to obtain unity gain.
- \( Lc \) might be also a dynamic system, e.g. a transfer function (2DOF controller).
- When comparing to the transfer-function of the process

\[ H(z) = C(zI - \Phi)^{-1} \Gamma \]

it can be seen that the zeros are not altered by state-feedback (see controllable canonical form).

- The rejection of disturbances can be handled as before.
2nd Approach

- Reference model:

\[
\begin{align*}
    x_m[k + 1] &= \Phi_m x_m[k] + \Gamma_m u_c[k] \\
    y_m[k] &= C_m x_m[k]
\end{align*}
\]

- Control law:

\[
u[k] = L(x_m[k] - \hat{x}[k|k - 1]) + u_{ff}[k]
\]

- \(x_m[k]\): desired state (coordinate systems must be compatible)
- \(u_{ff}\): control signal that gives desired output when applied to the open-loop system.
- feedback: \(L(x_m[k] - \hat{x}[k|k - 1])\), feedforward signal: \(u_{ff}\)
- Generation of the feedforward signal:

\[
    u_{ff}[k] = \frac{H_m(q)}{H(q)} u_c[k]
\]
• $H_m$, the transfer-function of the reference model,
  – must be stable,
  – the pole excess of the reference model must be not less than the pole excess of the process, and
  – unstable process zeros must also be zeros of the mode.

• If the order of the reference model and process are the same, a typical choice is:

\[
H(z) = \frac{B(z)}{A(z)}, \quad H_m(z) = \frac{\lambda B(z)}{A_m(z)}
\]