Exercise 7: Stability of the standard control loop

Pre-assignment:

1. Repeat the stability criteria for multi-input multi-output (MIMO) systems from the lecture.
2. Solve Exercise 7.1 a).

Exercise 7.1 (Chen-Hsu theorem)
Consider the plant $G(s)$ and the controller $K(s)$ of a standard control loop given by

$$G(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{s-1} & \frac{2}{s+1} \end{bmatrix} \quad \text{and} \quad K(s) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}, \quad k_1, k_2 > 0.$$

a) Derive the open-loop pole polynomial.

b) Determine the closed-loop pole polynomial using the Chen-Hsu theorem.

c) For which control parameters $k_1$ and $k_2$ is the closed loop asymptotically stable?

Exercise 7.2 (Comparison of stability criteria)
Consider plant $G(s)$ and controller $K(s)$ of a standard control loop given by the following matrices.

a) $K_a(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ 1 & \frac{1}{s+0.1} \end{bmatrix} \quad G_a(s) = \begin{bmatrix} \frac{s}{s+2} & \frac{1}{s+2} \\ 1 & \frac{1}{s^2-1} \end{bmatrix}$

b) $K_b(s) = \begin{bmatrix} 10 & \frac{1}{s+2} \\ 0 & \frac{10}{(s+2)(s+1)} \end{bmatrix} \quad G_b(s) = \begin{bmatrix} \frac{s-0.5}{s+10} & \frac{1}{s+10} \\ 1 & \frac{1}{s+10} \end{bmatrix}$

c) $K_c(s) = \begin{bmatrix} 10 & 0 \\ 0 & \frac{10}{(s+2)(s+1)} \end{bmatrix} \quad G_c(s) = \begin{bmatrix} \frac{1}{(s+2)(s+4)} & \frac{1}{s+4} \\ \frac{1}{s+4} & \frac{(s+2)^2}{(s+2)(s+4)} \end{bmatrix}$

d) $K_d(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s+0.5}{(s+2)(s+1)} \\ \frac{s+4}{(s+2)(s+1)} & \frac{1}{s+2} \end{bmatrix} \quad G_d(s) = \begin{bmatrix} \frac{1}{s+2}(s-4) & \frac{1}{s+4} \\ \frac{1}{s+4} & \frac{(s+2)^2}{s+5} \end{bmatrix}$

Address the following subtasks for each pair of plant and controller.

(i) The Nyquist plots in Figures show the determinants of the return difference matrices $\Gamma_1(s) = \det(I + Q(s)), \quad s \in \mathbb{N}_1$ with $Q(s) = G(s)K(s)$. Decide whether the closed-loop system is asymptotically stable using the Nyquist criterion for MIMO systems. Justify your statement.

1The pre-assignment should be addressed before and brought to the corresponding exercise.
(ii) The Nyquist plots of the eigenvalues $\lambda_{Q,i}(s)$, $s \in \mathbb{N}_1$, $i \in 1,2$ of the open-loop transfer function matrices $Q(s) = G(s)K(s)$ are shown in Figures 5–12. Decide whether the closed-loop system is asymptotically stable using the appropriate version of the Nyquist criterion. Justify your statement.
(iii) If appropriate, assess the asymptotic stability of the control loops using the Small-gain theorem. Compare the findings with the results from subtasks (i) and (ii).

(iv) The Nyquist plots of the diagonal elements $q_{ii}(s), s \in \mathcal{H}, i \in 1,2$ of the open-loop transfer function matrix $Q(s) = G(s)K(s)$ are shown together with generalised Gershgorin disks (gGds) in Figures 13-20. Can asymptotic closed-loop stability be assessed via gGds? If yes, analyse the stability of the control loops using gGds and compare the findings to the results from subtasks (i) and (ii).
Fig. 13: a) Nyquist plot of $q_{a,11}$ with gGds.

Fig. 14: a) Nyquist plot of $q_{a,22}$ with gGds.

Fig. 15: b) Nyquist plot of $q_{b,11}$ with gGds.

Fig. 16: b) Nyquist plot of $q_{b,22}$ with gGds.

Fig. 17: c) Nyquist plot of $q_{c,11}$ with gGds.

Fig. 18: c) Nyquist plot of $q_{c,22}$ with gGds.

Fig. 19: d) Nyquist plot of $q_{d,11}$ with gGds.

Fig. 20: d) Nyquist plot of $q_{d,22}$ with gGds.
Exercise 7.3 (Comparison of stability criteria for systems with variable controller gain)
Consider a standard control loop with plant $G(s)$ and controller $K(s)$ given by

$$G(s) = \begin{bmatrix} s-1 & s \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix} \quad \text{and} \quad K(s) = \begin{bmatrix} k \\ 0 \\ 0 \\ k \end{bmatrix}, \quad k \in \mathbb{R}, \quad k > 0.$$ 

a) Analyse the Nyquist plots in Figures 21–23 that show the determinants of the return difference matrix $\Gamma_1(s) = \det(I + Q(s)), \quad s \in \mathbb{N}_1$ with $Q(s) = G(s)K(s)$. Decide for which values of $k$ the closed-loop system is asymptotically stable using the Nyquist criterion for MIMO systems.

Fig. 21: Nyquist plot of $\Gamma_1(s), k = 0.8$.

Fig. 22: Nyquist plot of $\Gamma_1(s), k = 1.25$.

Fig. 23: Nyquist plot of $\Gamma_1(s), k = 2.5$.

b) Determine the pole polynomial of the closed loop using the Chen-Hsu theorem. For which control parameters $k$ is the closed-loop system asymptotically stable?

c) The Nyquist plots of the eigenvalues $\lambda_{Q_1}(s)$ und $\lambda_{Q_2}(s)$ of the open-loop transfer function matrix $Q(s) = G(s)K(s)$ are shown in Figures 24 and 25 for $s \in \mathbb{N}_1$ and $k = 1$. Decide for which values of $k$ the closed-loop system is asymptotically stable using the appropriate version of the Nyquist criterion.

d) If applicable, assess the stability of the control loops using the Small-gain theorem. Compare the findings with the results from subtask c).
Exercise 7.4 (Bonus question: Nyquist criterion with poles on imaginary axis)
Consider standard control loops with the following open-loop transfer function matrices.

a) \( G(s)K(s) = \begin{bmatrix} \frac{1}{s} & 1 \\ 1 & 1 \end{bmatrix} \)

b) \( G(s)K(s) = \begin{bmatrix} \frac{1}{s^2} & 1 \\ \frac{1}{s} & 1 \end{bmatrix} \)

The Nyquist plots of the determinants of the return difference matrix \( \Gamma_1(j\omega) = \det(I + Q(j\omega)) \) for \( \epsilon \leq \omega < \infty \) with \( 0 < \epsilon \ll 1 \) are shown in Figures 26 and 27. Decide whether the closed-loop system is asymptotically stable using the Nyquist criterion.
Exercise 7.5 (Generalised diagonal dominance of return difference matrix)
Consider a plant of the form
\[ G(s) = \begin{bmatrix} \frac{s+4}{s+1} & \frac{s+2}{s+1} \\ \frac{1}{s+4} & \frac{1}{s+2} \end{bmatrix} \]
that shall be controlled using
\[ K(s) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}, \quad k_1, k_2 > 0. \]
a) Decide for which control parameters \( k_1 \) and \( k_2 \) the return difference matrix is generalised diagonally dominant. Therefore, first calculate the Perron-Frobenius eigenvalue of the comparison matrix of \( Q(s) = G(s)K(s) \) as a function of \( \omega \) and find a small upper bound. Then, decide for which values of the control parameters \( k_1 \) and \( k_2 \) the generalised Gershgorin bands cover the critical point.
b) What can you say about the stability of the closed-loop system?

Exercise 7.6 (Decoupling)
In Exercise 7.5 you could see that it is beneficial in the design of a MIMO controller if the Perron-Frobenius eigenvalue \( \lambda_{PF}(C_{c}(Q(s))) \) is small. In a best-case scenario, the cross-coupling \( G_{12} \) and \( G_{21} \) can be neglected and the control design of a MIMO controller can be reduced to the design of multiple single-input single-output (SISO) controllers. This is schematically illustrated in Figures 28 and 29. Here, the achievable performance of the closed-loop system increases with decreasing cross-coupling. The plant is exactly decoupled, i.e., \( G_{12} = 0 \) or \( G_{21} = 0 \), if the Perron-Frobenius eigenvalue of the comparison matrix is zero for all \( \omega \).

In practice, there are different ways to reduce strong cross-coupling of MIMO systems using a decoupling matrix \( K_c \). In this exercise, a simple decoupling matrix called permutation matrix

\[ K_c = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \]

2 The design of such decoupling matrices is not part of this lecture. More information on this topic can be found, e.g., in the book “Mehrgroßenregelung im Frequenzbereich” by Jörg Raisch on pages 247ff.
shall be employed. The goal of this matrix is to change the mapping of inputs and outputs such that the Perron-Frobenius eigenvalue of the comparison matrix of $[I]$ is small.

$$G(s) = \begin{bmatrix} \frac{s+1}{5(s+3)} & \frac{2}{s+2} \\ \frac{12}{s+3} & \frac{s}{5(s+1)} \end{bmatrix} \tag{1}$$

a) Plot the Perron-Frobenius eigenvalue of the comparison matrix $C_c(G(j\omega))$ in Python for $\omega \in (0,100]$.

b) Multiply $G(s)$ by a permutation matrix $K_c$ that exchanges the columns of $G(s)$ and thereby alternatively maps inputs and outputs. Then, plot the Perron-Frobenius eigenvalue of the comparison matrix $C_c(G(j\omega)K_c)$ for $\omega \in (0,100]$ in Python.

c) Decide which mapping of input and output variables is most suitable for a simplified control synthesis. Justify your statement.

d) **Bonus question:** Try to solve subtask c) without Python. First, analyse the limit value of the Perron-Frobenius eigenvalue of the comparison matrix $C_c(G(j\omega))$ for large and small values of $\omega$. Then, consider that an arbitrary diagonal controller of the form $K(s) = \text{diag}\{k_1(s), k_2(s)\}$ with integrating or proportional behaviour (i.e., $\lim_{s \to 0} k_1(s) \neq 0$ and $\lim_{s \to 0} k_2(s) \neq 0$) is used. What can you say about the diameters of the generalised Gershgorin disk of the open-loop transfer function matrix?

*Remark:* Recall that the generalised Gershgorin bands need to be narrow such that they do not cover the critical point.

### Some Python Control Systems and Numpy commands

- `myTf = control.tf(...)`: Create transfer function
- `myTf(1j)`: Evaluate transfer function $myTf$ for $s = 1j$.
- `control.minreal`: Minimal realisation of transfer function
- `numpy.logspac`: Define vector with logarithmic distances
- `numpy.sqrt`: Square root
- `numpy.abs`: Absolute value of complex number
- `matplotlib.pyplot.loglog`: Semilogarithmic plot