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Asynchronous l-complete approximations☆

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ABSTRACT

This paper extends the l-complete approximation method developed for time invariant systems to a larger system class, ensuring that the resulting approximation can be realized by a finite state machine. To derive the new abstraction method, called asynchronous l-complete approximation, an asynchronous version of the well-known concepts of state property, memory span and l-completeness is introduced, extending the behavioral systems theory in a consistent way.

1. Introduction

Real life control problems for large scale systems are very challenging due to numerous interactions between different components and usually tight performance requirements. One way to reduce the complexity of those control problems is to introduce different control layers using a well defined abstraction of the plant. Usually, the top control layer will enforce high level specifications, such as interconnection or safety requirements, typically expressible by regular languages. With this specification type supervisory control theory (SCT) [1,2] can be used to synthesize a correct by design control system if the abstracted plant model can be represented by a regular language as well.

Using this well known result, many abstraction techniques, e.g. [3–10], have been developed to generate a regular language representation of the plant model. The approach originally introduced by Moor and Raisch [3], called l-complete approximation, is distinct in two ways. (i) The accuracy of the abstracted system can be adjusted during construction without adjusting the external signal space. This property was for example used in the recent publications [11–13]. (ii) The behavioral framework [14,15] is used to model the plant. If the external signal space is finite and signals evolve on the non-negative discrete time axis $\mathbb{N}_0$, the plant behavior is a so called ω-language. Even though SCT cannot be directly applied to ω-languages, it was shown in [3,16] that for ω-languages realizable by finite state machines (FSM), a variant of SCT can be used to synthesize a minimally restrictive controller for specifications representable by the closure of a regular language (for details, see [16] and the references therein).

In [3] and subsequent papers, l-complete approximations were only defined for time invariant systems evolving on $\mathbb{N}_0$, i.e., systems that are invariant w.r.t. the backward time shift of signals. For systems not having the latter property, [17, p. 51] proposes a pragmatic extension.

As pointed out in [17, p. 44], for systems evolving on $\mathbb{N}_0$, the l-completeness property for time invariant systems used in [3,17] is slightly stronger than the original definition by J.C. Willems [14]. This implies that the strongest l-complete approximation suggested in [3] is also l-complete in the sense of [14], but not necessarily the *strongest* l-complete approximation in the sense of [14].

To resolve this inconsistency and to consider a larger system class, we extend the construction of strongest l-complete approximations to not necessarily time invariant systems, and ensure that these approximations can still be realized by FSMs.

As a first step, in Sections 3 and 4, we introduce a straightforward extension of the existing approximation method to not necessarily time invariant dynamical systems, ensuring l-completeness in the sense of [14]. We show in Section 5 that the constructed abstractions do generally not allow for an FSM realization since they require a time dependent next state relation. Intuitively, a system is realizable by a state machine if it allows

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for concatenation of state trajectories that reach the same state asynchronously (i.e., at different times), as used in the context of state maps by Julius and van der Schaft [18,19]. To emphasize that this property does not imply and is not implied by the time invariance property of behavioral systems, we call it *asynchronous state property* and formalize it in Section 6. Then we can introduce an *asynchronous l-completeness* property, since the state and the *l*-completeness property are strongly related. This leads to a new approximation technique introduced in Section 7, which is referred to as *asynchronous l-complete approximation* and which ensures that the resulting abstraction can be realized by an FSM.

2. Preliminaries

In the behavioral framework (e.g., [15]) a *dynamical system* is given by $\Sigma = (T, W, \mathcal{B})$, consisting of the time axis T (in this paper: $T = \mathbb{N}_0$), the signal space W and the behavior of the system $\mathcal{B} \subseteq W^T$, where $W^T := \{w \mid w : T \to W\}$ is the set of all *signals* taking values in W. Let $\mathcal{I}$ be a bounded interval on $\mathbb{N}_0$, then $W^{\mathcal{I}} := \{w \mid w : \mathcal{I} \to W\}$ is the set of *signals on* $\mathcal{I}$ taking values in W. Furthermore, $w|_{\mathcal{I}}$ is the *restriction* of the map $w : \mathbb{N}_0 \to W$ to the domain $\mathcal{I}$. To keep notation compact, we will not distinguish between $w|_{\mathcal{I}} \in W^{\mathcal{I}}$ and $w|_{\mathcal{I}} \in W^{|\mathcal{I}|}$, i.e., we disregard absolute time information when restricting a signal to an interval $\mathcal{I}$. We say that $w|_{\mathcal{I}}$ has length $|w|_{\mathcal{I}}|_L = |\mathcal{I}| = t_2 - t_1 + 1$. $\mathcal{B}|_{\mathcal{I}} \subseteq W^{\mathcal{I}}$ denotes the restriction of all signals in $\mathcal{B}$ to $\mathcal{I}$ and we define[1] $\forall t_1, t_2 \in \mathbb{N}_0,\ t_1 < t_2 \,.\, w\,|_{[t_2,t_1]} = \lambda$, where λ denotes the *empty string* with $|\lambda|_L = 0$. Now let $W = W_1 \times W_2$ be a product space. Then the *projection* of a signal $w \in W^{\mathbb{N}_0}$ to W_1 is given by $\pi_{W_1}(w) := \{w_1 \in W_1^{\mathbb{N}_0} \mid \exists w_2 \in W_2^{\mathbb{N}_0} \,.\, w = (w_1, w_2)\}$ and $\pi_{W_1}(\mathcal{B})$ denotes the projection of all signals in the behavior to W_1. Given two signals $w_1, w_2 \in W^{\mathbb{N}_0}$ and two time instants $t_1, t_2 \in \mathbb{N}_0$, the *concatenation* $w_3 = w_1 \wedge_{t_2}^{t_1} w_2$ is given by

$$\forall t \in \mathbb{N}_0 \,.\, w_3(t) = \begin{cases} w_1(t), & t < t_1 \\ w_2(t - t_1 + t_2), & t \geq t_1, \end{cases} \tag{1}$$

where we denote $\cdot \wedge_t^t \cdot$ by $\cdot \wedge_t \cdot$. Furthermore, the concatenation of their restrictions $w_1' = w_1\,|_{[0,t_1]}$ and $w_2' = w_2\,|_{[0,t_2]}$ is defined as $w_1' \cdot w_2' := \left(w_1 \wedge_0^{t_1+1} w_2\right)|_{[0,t_1+t_2+1]}$. This corresponds to the standard concatenation of finite strings. Following [15, Definition II.3], we define the *backward shift operator* σ^t s.t. $\forall t, k \in \mathbb{N}_0 \,.\, (\sigma^t f)(k) := f(k+t)$ and say that Σ is *time invariant* if $\sigma\mathcal{B} \subseteq \mathcal{B}$. We call Σ *strictly time invariant* if $\sigma\mathcal{B} = \mathcal{B}$.

3. *l*-completeness and *l*-complete approximation

When reasoning about systems with infinite time axis one has to distinguish between local and eventuality properties. Local properties can be evaluated on a finite time interval whereas eventuality properties can only be evaluated after infinite time. Systems whose behavior can be fully described by local properties are called complete [15, Definition II.4]; formally, Σ is said to be *complete* if

$$\left(\forall t_1, t_2 \in \mathbb{N}_0, t_1 \leq t_2 \,.\, w\,|_{[t_1,t_2]} \in \mathcal{B}\,|_{[t_1,t_2]}\right) \Leftrightarrow w \in \mathcal{B}. \tag{2}$$

It is easy to show that (2) is equivalent to

$$\left(\forall \tau \in \mathbb{N}_0 \,.\, w\,|_{[0,\tau]} \in \mathcal{B}\,|_{[0,\tau]}\right) \Leftrightarrow w \in \mathcal{B}, \tag{3}$$

which is also known as ω-closedness [20].

[1] Throughout this paper we use the notation "$\forall \cdot \,.\, \cdot$", meaning that all statements after the dot hold for all variables in front of the dot. "$\exists \cdot \,.\, \cdot$" is interpreted analogously.

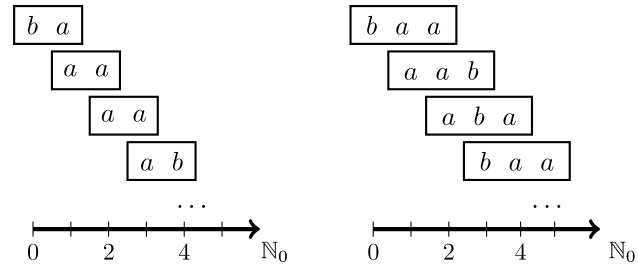

Fig. 1. Domino game for $l = 1$ (left) and $l = 2$ (right) in Example 1.

In the special case where the behavior can be fully described by local properties evaluated on time intervals of length $l + 1$, $l \in \mathbb{N}$, the system is called *l-complete* [14, p. 184], formally

$$\left(\forall t \in \mathbb{N}_0 \,.\, w\,|_{[t,t+l]} \in \mathcal{B}\,|_{[t,t+l]}\right) \Leftrightarrow w \in \mathcal{B}. \tag{4}$$

To generate some intuition for the *l*-completeness property, we define $\mathcal{D}_{l+1} = \bigcup_{t' \in \mathbb{N}_0} \mathcal{B}\,|_{[t',t'+l]}$ to be the set of all finite strings representing the restriction of admissible signals to a time interval of length $l + 1$. Now consider the following gedankenexperiment: assume playing a sophisticated domino game where $\mathcal{D}_{l+1}$ is the set of dominos. Pick the first domino from the set $\mathcal{B}\,|_{[0,l]}$ and append one domino from the set $\mathcal{B}\,|_{[1,1+l]}$ if the last l symbols of the first domino are equivalent to the first l symbols of the second domino. Playing the domino game arbitrarily long and with all possible initial conditions and domino combinations, we get the set $\mathcal{B}^l$ containing all signals that satisfy the left side of (4). If the system is *l*-complete we have $\mathcal{B} = \mathcal{B}^l$, emphasizing that all valid signals can be fully described by a local property.

Example 1. Consider the system

$$\begin{aligned} &\Sigma = (\mathbb{N}_0, W, \mathcal{B}) \quad s.t. \\ &\mathcal{B} = \{aaab(aab)^\omega, aab(aab)^\omega, ab(aab)^\omega, b(aab)^\omega\}, \end{aligned} \tag{5}$$

where $(\cdot)^\omega$ denotes the infinite repetition of the respective string. Observe that Σ is time invariant, but not strictly time invariant, since

$$\begin{aligned} &aaab(aab)^\omega \notin \sigma\mathcal{B} \\ &\quad = \{aab(aab)^\omega, ab(aab)^\omega, b(aab)^\omega, (aab)^\omega\}. \end{aligned}$$

Using $l = 1$ we get the domino set

$$\forall t \in \mathbb{N}_0 \,.\, \mathcal{B}\,|_{[t,t+1]} = \mathcal{B}\,|_{[0,1]} = \{aa, ab, ba\}. \tag{6}$$

As depicted in Fig. 1 (left), we can start the domino game with the piece *ba* and append a piece that starts with an *a*, e.g., *aa*. Observe that the signal constructed in Fig. 1 (left), i.e., $w = baaab \ldots$, is not allowed in (5) since not more than two sequential *a*'s can occur for $t > 0$. However, we can of cause construct all signals $w \in \mathcal{B}$ using the outlined domino game. This implies that (i) the system Σ in (5) is not 1-complete and (ii) the domino game constructs a behavior $\mathcal{B}^1$ that is larger than the one in (5), i.e., $\mathcal{B}^1 \supset \mathcal{B}$. Now, increasing l to $l = 2$ gives the following set of domino pieces

$$\begin{aligned} &\mathcal{B}\,|_{[0,2]} = \{aaa, aab, aba, baa\}, \\ &\forall t > 0 \,.\, \mathcal{B}\,|_{[t,t+2]} = \mathcal{B}\,|_{[1,3]} = \{aab, aba, baa\}. \end{aligned} \tag{7}$$

Playing the domino game with these sets results, for example, in the signal depicted in Fig. 1 (right), where always two symbols are required to match. Observe that after the first piece we are only allowed to pick from the set $\mathcal{B}\,|_{[1,3]}$. This prevents the occurrence of more than two sequential *a*'s since the domino *aaa* cannot be attached. We get $\mathcal{B}^2 = \mathcal{B}$, i.e., the system Σ in (5) is 2-complete. ◁

As a special case it can be shown that the behavior of an l-complete system Σ can by fully described by the initial signal pieces $\mathcal{B}|_{[0,l]}$ if Σ is strictly time invariant.

Lemma 1. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a strictly time invariant dynamical system and $l \in \mathbb{N}$. Then Σ is l-complete iff*

$$\left(\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[0,l]}\right) \Leftrightarrow w \in \mathcal{B}. \tag{8}$$

Proof. If Σ is strictly time invariant, i.e., $\sigma\mathcal{B} = \mathcal{B}$, then $\forall t \in \mathbb{N}_0 \,.\, \sigma^t\mathcal{B} = \mathcal{B}$, hence $\forall t \in \mathbb{N}_0 \,.\, \mathcal{B}|_{[t,t+l]} = \mathcal{B}|_{[0,l]}$. Therefore, (8) and (4) are equivalent. □

Remark 1. For systems that are time invariant but not strictly time invariant, $\exists t \in \mathbb{N}_0 \,.\, \mathcal{B}|_{[t,t+l]} \subset \mathcal{B}|_{[0,l]}$, implying that in this case (4) and (8) are not equivalent. Therefore, the definition of l-completeness via (8), as used in [3, Definition 8] and subsequent papers, does not coincide with the definition of l-completeness via (4), as originally used by J.C. Willems [14, Section 1.4.1]. In particular, an l-complete system in the sense of [3] (i.e., a system satisfying (8)) is also l-complete in the sense of [14] (i.e., it also satisfies (4)) but the inverse implication does not hold in general. To see this, suppose that $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ is l-complete in the sense of [3]. Then, obviously,

$$\left(\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[0,l]}\right) \Rightarrow w \in \mathcal{B}$$

and $\forall t \in \mathbb{N}_0 \,.\, \mathcal{B}|_{[t,t+l]} \subseteq \mathcal{B}|_{[0,l]}$. It follows that

$$\left(\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[t,t+l]}\right) \Rightarrow w \in \mathcal{B},$$

which implies l-completeness in the sense of [14]. Therefore, l-completeness in the sense of [3] is a stronger property than l-completeness in the sense of [14].

The set $\mathcal{B}^l$ generated in the outlined domino game also matches the behavior of the system Σ, if Σ is r-complete with $r \leq l$, since using larger dominos cannot lead to a richer behavior. Furthermore, as already shown in Example 1, we will always get $\mathcal{B}^l \supseteq \mathcal{B}$ even if the system is not complete at all, since using less information in the domino game generates more freedom in constructing signals. Formalizing this idea, following [3, Definition 9] we say that $\Sigma^l = (\mathbb{N}_0, W, \mathcal{B}^l)$ is an *l-complete approximation* of $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$, if (i) Σ^l is l-complete and (ii) $\mathcal{B}^l \supseteq \mathcal{B}$. Furthermore, $\Sigma^{l\Uparrow} = (\mathbb{N}_0, W, \mathcal{B}^{l\Uparrow})$ is the *strongest l-complete approximation* of Σ, if (i) $\Sigma^{l\Uparrow}$ is an l-complete approximation of Σ and (ii) for any l-complete approximation $\Sigma' = (\mathbb{N}_0, W, \mathcal{B}')$ of Σ it holds that $\mathcal{B}^{l\Uparrow} \subseteq \mathcal{B}'$.

Remark 2. As an immediate consequence of Remark 1, l-complete approximations introduced in [3] are always l-complete approximations as defined above, but the reverse implication does generally not hold.

Generalizing the results in [3, Proposition 10] to the l-completeness definition in (4) shows that the behavior $\mathcal{B}^l$ constructed in the outlined domino game is the behavior of the strongest l-complete approximation, $\mathcal{B}^{l\Uparrow}$.

Theorem 1. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a dynamical system. Then the unique strongest l-complete approximation of Σ is given by $\Sigma^{l\Uparrow} = (\mathbb{N}_0, W, \mathcal{B}^{l\Uparrow})$, with*

$$\mathcal{B}^{l\Uparrow} := \left\{w \in W^{\mathbb{N}_0} \middle| \forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[t,t+l]} \,.\right\} \tag{9}$$

Furthermore, if Σ is strictly time invariant then

$$\mathcal{B}^{l\Uparrow} = \left\{w \in W^{\mathbb{N}_0} \middle| \forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[0,l]}\right\}. \tag{10}$$

Proof. (i) $\Sigma^{l\Uparrow}$ is l-complete as (9) implies $\mathcal{B}^{l\Uparrow}|_{[t,t+l]} = \mathcal{B}|_{[t,t+l]}$, hence $w \in \mathcal{B}^{l\Uparrow} \Leftrightarrow \left(\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}^{l\Uparrow}|_{[t,t+l]}\right)$.

(ii) $\mathcal{B} \subseteq \mathcal{B}^{l\Uparrow}$ holds, as $w \in \mathcal{B}$ implies $\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[t,t+l]}$, hence $w \in \mathcal{B}^{l\Uparrow}$ from (9).

(iii) For any l-complete approximation $\Sigma' = (\mathbb{N}_0, W, \mathcal{B}')$ of Σ the inclusion $\mathcal{B} \subseteq \mathcal{B}'$ and therefore $\mathcal{B}|_{[t,t+l]} \subseteq \mathcal{B}'|_{[t,t+l]}$ holds. Hence, using (9), $w \in \mathcal{B}^{l\Uparrow}$ implies $\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}'|_{[t,t+l]}$ and therefore $w \in \mathcal{B}'$ since Σ' is l-complete.

Now (i)–(iii) imply that $\Sigma^{l\Uparrow}$ is a strongest l-complete approximation. Finally, $\Sigma^{l\Uparrow}$ is unique as (iii) implies that $\mathcal{B}^{l\Uparrow}$ is the unique smallest element of the set $\{\mathcal{B}'\}$ containing the behaviors of all l-complete approximations $\Sigma' = (\mathbb{N}_0, W, \mathcal{B}')$ of Σ.

The second part of the theorem follows directly from (8) in Lemma 1. □

Example 2. As a consequence of Theorem 1, the behaviors $\mathcal{B}^1$ and $\mathcal{B}^2$ constructed in Example 1 characterize the strongest 1-complete and the strongest 2-complete approximation of the system in (5), respectively. ◁

Remark 3. The strongest l-complete approximation for a time invariant system Σ using the l-completeness property in the sense of [3] was shown to be characterized by the behavior $\widetilde{\mathcal{B}} = \left\{w \in W^{\mathbb{N}_0} \middle| \forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[0,l]}\right\}$. Then, since time invariance implies $\forall t \in \mathbb{N}_0 \,.\, \mathcal{B}|_{[t,t+l]} \subseteq \mathcal{B}|_{[0,l]}$, we have $\mathcal{B}^{l\Uparrow} \subseteq \widetilde{\mathcal{B}}$. Therefore, the strongest l-complete approximation as defined in Theorem 1 is a tighter approximation than the strongest l-complete approximation defined in [3].

4. State space systems

To represent a behavior, internal variables can be useful. Following [14, Definition 1.2], a *dynamical system with internal signal space X* is defined by $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ with $\mathcal{B}_S \subseteq (W \times X)^{\mathbb{N}_0}$. The internal variables are called states, if the axiom of state holds, i.e., all relevant information from the past and present necessary to decide on the possible future evolution of the system is captured by the current value of the internal variable. Formally, a system $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ is a *state space dynamical system* [14, p. 185], if

$$\forall\, (w_1, x_1), (w_2, x_2) \in \mathcal{B}_S, t \in \mathbb{N}_0 \,.\, \left(x_1(t) = x_2(t) \Rightarrow (w_1, x_1) \wedge_t (w_2, x_2) \in \mathcal{B}_S\right), \tag{11}$$

and Σ_S is a *state space representation* of $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ if $\pi_W(\mathcal{B}_S) = \mathcal{B}$. Recalling the gedankenexperiment in Section 3, all necessary information to determine the future evolution (i.e., the next feasible domino) is captured in the last l symbols. Systems which exhibit this property are said to have *memory span l* [14, p. 184], formally

$$\forall\, w_1, w_2 \in \mathcal{B}, t \in \mathbb{N}_0 \,.\, \left(w_1|_{[t,t+l-1]} = w_2|_{[t,t+l-1]} \Rightarrow w_1 \wedge_t w_2 \in \mathcal{B}\right). \tag{12}$$

It follows intuitively from the gedankenexperiment that any l-complete system has memory span l. However, the reverse implication only holds if the system is complete to ensure that its behavior can be fully described by a local property such as a finite memory span. This statement was proven in [14, Proposition 1.1] for time axis $\mathbb{Z}$ and trivially specializes to $\mathbb{N}_0$.

Proposition 1. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a dynamical system and $l \in \mathbb{N}$. Then*

$$\Sigma \text{ is } l\text{-complete} \Leftrightarrow \begin{pmatrix} \Sigma \text{ is complete} \\ \wedge \Sigma \text{ has memory span } l \end{pmatrix} \tag{13}$$

Proof. "$\Rightarrow$" Obviously l-completeness implies completeness (from (3) and (4)). To show that l-completeness of Σ implies (12), fix $w_1, w_2 \in \mathcal{B}, t \in \mathbb{N}_0$ s.t. $w_1|_{[t,t+l-1]} = w_2|_{[t,t+l-1]}$ and show $w = w_1 \wedge_t w_2 \in \mathcal{B}$.

As $w_1|_{[t,t+l-1]} = w_2|_{[t,t+l-1]}$ it holds that

$$w|_{[t',t'+l]} = \begin{cases} w_1|_{[t',t'+l]}, & t' < t \\ w_2|_{[t',t'+l]}, & t' \geq t, \end{cases}$$

hence $\forall t' \in \mathbb{N}_0 \,.\, w|_{[t',t'+l]} \in \mathcal{B}|_{[t',t'+l]}$. With l-completeness of Σ it follows that $w \in \mathcal{B}$ (from (4)).

"$\Leftarrow$" To show that (4) holds for a complete system Σ with memory span l, fix $w \in W^{\mathbb{N}_0}$ s.t. the left hand side of (4) holds and show that $w \in \mathcal{B}$ follows. Observe that the left hand side of (4) implies

$$\exists w_0, w_1 \in \mathcal{B} \,.\, \begin{pmatrix} w_0|_{[0,l]} = w|_{[0,l]} \\ \wedge w_1|_{[1,l+1]} = w|_{[1,l+1]} \\ \wedge w_0|_{[1,l]} = w_1|_{[1,l]} \end{pmatrix}. \tag{14}$$

As Σ has memory span l, the last line in (14) implies $w_0 \wedge_1 w_1 \in \mathcal{B}$, hence $(w_0 \wedge_1 w_1)|_{[0,l+1]} = w|_{[0,l+1]} \in \mathcal{B}|_{[0,l+1]}$. Iteratively applying this procedure therefore yields $\forall \tau \in \mathbb{N}_0 \,.\, w|_{[0,\tau]} \in \mathcal{B}|_{[0,\tau]}$ implying $w \in \mathcal{B}$ as Σ is complete. $\square$

Now we can conclude that (i) every l-complete system has memory span l, (ii) the state property implies that $\Sigma_x = (\mathbb{N}_0, X, \pi_X(\mathcal{B}_S))$ has memory span one and (iii) a straightforward choice for the state space of systems with memory span l is given by the set of admissible strings[2] of length l. Considering also the fact that for the first l time steps we can only memorize the symbols already seen, we can generalize the construction of a state space representation given in [3, p. 6].

Lemma 2. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a system with memory span l. Furthermore, let*

$$X := \left(\bigcup_{r \in [0,l-1]} \mathcal{B}|_{[0,r-1]} \right) \cup \left(\bigcup_{t \in \mathbb{N}_0} \mathcal{B}|_{[t,t+l-1]} \right)$$

and let $\mathcal{B}_S \subseteq (W \times X)^{\mathbb{N}_0}$ s.t. $(w, x) \in \mathcal{B}_S$ iff

$$x(t) = \begin{cases} w|_{[0,t-1]} & 0 \leq t < l \\ w|_{[t-l,t-1]} & t \geq l \end{cases} \tag{15}$$

and $w \in \mathcal{B}$. Then $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ is a state space representation of Σ.

Proof. $\pi_W(\mathcal{B}_S) = \mathcal{B}$ holds by construction. To show (11), pick $(w_1, x_1), (w_2, x_2) \in \mathcal{B}_S$ and $t' \in \mathbb{N}_0$ s.t. $x_1(t') = x_2(t')$ and show $(w, x) = (w_1, x_1) \wedge_{t'} (w_2, x_2) \in \mathcal{B}_S$: observe that

$$\begin{aligned} x_1(t') &= w_1|_{[\max\{0,t'-l\},t'-1]} \\ &= w_2|_{[\max\{0,t'-l\},t'-1]} = x_2(t'). \end{aligned} \tag{16}$$

This implies for $t' < l$ that $w = w_1 \wedge_{t'} w_2 = w_2 \in \mathcal{B}$. From Σ having memory span l (16) implies[3] $w = w_1 \wedge_{t'} w_2 \in \mathcal{B}$ for $t' \geq l$. Now remember that (15) holds for $(w_1, x_1), (w_2, x_2) \in \mathcal{B}_S$. Therefore, $x = x_1 \wedge_{t'} x_2$ implies that for all $t \in \mathbb{N}_0$

$$x(t) = \begin{cases} w_1|_{[0,t-1]} & (t \leq t') \wedge (t < l) \\ w_1|_{[t-l,t-1]} & (t \leq t') \wedge (t \geq l) \\ w_1|_{[0,t'-1]} \cdot w_2|_{[t',t-1]} & (t' < t < t'+l) \wedge (t < l) \\ w_1|_{[t-l,t'-1]} \cdot w_2|_{[t',t-1]} & (t' < t < t'+l) \wedge (t \geq l) \\ w_2|_{[t-l,t-1]} & (t \geq t'+l)\,. \end{cases}$$

Hence, with $w = w_1 \wedge_{t'} w_2 \in \mathcal{B}, x = x_1 \wedge_{t'} x_2$ satisfies (15), proving $(w, x) \in \mathcal{B}_S$. $\square$

Since the strongest l-complete approximation $\Sigma^{l\Uparrow} = (\mathbb{N}_0, W, \mathcal{B}^{l\Uparrow})$ of any dynamical system $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ is l-complete it follows from Proposition 1 that $\Sigma^{l\Uparrow}$ has memory span l and we can use Lemma 2 to construct a state space representation of $\Sigma^{l\Uparrow}$, denoted by $\Sigma_S^{l\Uparrow} = (\mathbb{N}_0, W, X, \mathcal{B}_S^{l\Uparrow})$. Note that the state space X constructed in Lemma 2 has finitely many elements if $|W| < \infty$.

Example 3. Recall that the system in Example 1 is 2-complete and (6) implies $\bigcup_{t \in \mathbb{N}_0} \mathcal{B}|_{[t,t+1]} = \{aa, ab, ba\}$. Adding the set $\bigcup_{r \in [0,1]} \mathcal{B}|_{[0,r-1]} = \{\lambda, a, b\}$, the state space defined in Lemma 2 for a state space representation of the system Σ in (5) (and its strongest 2-complete approximation) is given by $X = \{\lambda, a, b, aa, ab, ba\}$. Analogously, the state space representation of the strongest 1-complete approximation of Σ has state space $X = \{\lambda, a, b\}$. $\triangleleft$

5. Evolution laws

To capture the step-by-step evolution of a system, so called evolution laws can be used (see [14, Section 1.5]).

Definition 1. A *time dependent discrete time evolution law* is a tuple $\Sigma_\psi = (\mathbb{N}_0, W, X, \psi, X_0)$, where X is the state space, W is the signal space, $X_0 \subseteq X$ is the set of initial states and $\psi : \mathbb{N}_0 \to 2^{X \times W \times X}$ is a time dependent next state relation. Furthermore, the *behavior induced by* ψ is defined by

$$\mathcal{B}_\psi = \left\{ (w, x) \middle| \begin{pmatrix} x(0) \in X_0 \\ \wedge \forall t \in \mathbb{N}_0 \,.\, (x(t), w(t), x(t+1)) \in \psi(t) \end{pmatrix} \right\}. \tag{17}$$

Let $\delta := \bigcup_{t \in \mathbb{N}_0} \psi(t) \subseteq X \times W \times X$ be a time independent next state relation and

$$\mathcal{B}_\delta := \left\{ (w, x) \middle| \begin{pmatrix} x(0) \in X_0 \\ \wedge \forall t \in \mathbb{N}_0 \,.\, (x(t), w(t), x(t+1)) \in \delta \end{pmatrix} \right\} \tag{18}$$

the behavior induced by it. Then we call Σ_ψ time independent if $\mathcal{B}_\psi = \mathcal{B}_\delta$.

When Σ_ψ is time independent, the definition of Σ_ψ simplifies to the time independent discrete time evolution law $\Sigma_\delta = (\mathbb{N}_0, W, X, \delta, X_0)$ with induced behavior $\mathcal{B}_\delta$. Note that $\mathcal{B}_\delta$ is not necessarily time invariant in the (behavioral) sense of Section 2. Observe that for unrestricted initial states, i.e., $X_0 = X$, Σ_δ coincides with [14, Definition 1.4] and the behavior $\mathcal{B}_\delta$ becomes time invariant and coincides with the one given in [14, p. 189]. Furthermore, in the special case where X and W are finite, the time independent evolution law Σ_δ defines a *finite state machine* (FSM) $P = (X, W, \delta, X_0)$ as, e.g., used in [3, Definition 3].

Based on the behavior of a state space system we construct an evolution law reproducing its step-by-step evolution.

[2] In contrast to [3] this choice of the state space represents only the reachable part of W^l.

[3] Observe that, under the premises of (12), $w_1 \wedge_t w_2 = w_1 \wedge_{t+l} w_2$ in the right side of the implication in (12).

Definition 2. Let $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ be a state space dynamical system and $\Sigma_\psi = (\mathbb{N}_0, W, X, \psi, X_0)$ an evolution law s.t. $X_0 = \pi_X(\mathcal{B}_S)\,|_{[0,0]}$ and

$$\psi(t) := \left\{ (\xi, \omega, \xi') \,\middle|\, \exists (w, x) \in \mathcal{B}_S \,.\, \begin{pmatrix} x(t) = \xi \\ \wedge w(t) = \omega \\ \wedge x(t+1) = \xi' \end{pmatrix} \right\}. \tag{19}$$

Then Σ_ψ is the evolution law induced by Σ_S.

By construction, an evolution law Σ_ψ induces a behavior that can be determined by a local property, i.e., the next state relation. Therefore, the behavior $\mathcal{B}_\psi$ induced by Σ_ψ and the behavior $\mathcal{B}_S$ only coincide if Σ_S is complete. This is formally stated in the following theorem, extending [14, Theorem 1.1] to time variant systems with $T = \mathbb{N}_0$.

Theorem 2. *Let $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ be a state space dynamical system and $\Sigma_\psi = (\mathbb{N}_0, W, X, \psi, X_0)$ the evolution law induced by it. Then*

$$\left(\mathcal{B}_S = \mathcal{B}_\psi\right) \Leftrightarrow \Sigma_S \text{ is complete}. \tag{20}$$

Proof. Fix any $(w, x) \in \mathcal{B}_\psi$. Now using (19) in (17) implies

$$\exists\, (w_0, x_0), (w_1, x_1) \in \mathcal{B}_S \,.\, \begin{pmatrix} x_0|_{[0,1]} = x|_{[0,1]} \wedge w_0(0) = w(0) \\ \wedge x_1|_{[1,2]} = x|_{[1,2]} \wedge w_1(1) = w(1) \\ \wedge x_0(1) = x_1(1) \end{pmatrix}. \tag{21}$$

As Σ_S is a state space dynamical system, the last line in (21) implies $(w_0, x_0) \wedge_1 (w_1, x_1) \in \mathcal{B}_S$ (from (11)) hence $(w, x)\,|_{[0,1]} \in \mathcal{B}_S\,|_{[0,1]}$. Applying this argument iteratively gives $\mathcal{B}_\psi = \left\{(w, x) | \forall \tau \in \mathbb{N}_0 \,.\, (w, x)\,|_{[0,\tau]} \in \mathcal{B}_S\,|_{[0,\tau]}\right\}$. Therefore, using (3), obviously $\mathcal{B}_\psi = \mathcal{B}_S$ iff Σ_S is complete. □

Definition 3. Given the premises of Theorem 2, we say that Σ_S is realized by Σ_ψ if Σ_S is complete. Furthermore, we say that Σ_S is realized by the FSM $P = (X, W, \delta, X_0)$ with $\delta := \bigcup_{t \in \mathbb{N}_0} \psi(t)$ if Σ_S is complete, Σ_ψ is time independent and X and W are finite.

Recall that in the presented domino game (see Section 3), a transition from one state to another is represented by adding an allowed domino. However, the set of allowed dominos is time dependent since we have to pick from the subset $\mathcal{B}\,|_{[t,t+l]}$ of all dominos $\mathcal{D}_{l+1}$ at time t. This suggests that the behavior $\mathcal{B}_S$ of a state space representation Σ_S of the l-complete dynamical system Σ can be realized by a time dependent evolution law Σ_ψ. This is formalized in the following theorem, extending [3, Theorem 12] to l-complete dynamical systems in the sense of (4), including also time variant systems.

Theorem 3. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be an l-complete dynamical system and $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ its state space representation constructed in Lemma 2. Then Σ_S is realized by the evolution law $\Sigma_\psi = (\mathbb{N}_0, W, X, \psi, X_0)$ with $X_0 = \{\lambda\}$ and*

$$\begin{aligned} \psi(t) = &\left\{(\xi, \omega, \xi \cdot \omega) | t < l \wedge \xi \cdot \omega \in \mathcal{B}\,|_{[0,t]}\right\} \\ &\cup \left\{(\xi, \omega, \xi\,|_{[1,l-1]} \cdot \omega) | t \geq l \wedge \xi \cdot \omega \in \mathcal{B}\,|_{[t-l,t]}\right\}. \end{aligned} \tag{22}$$

Proof. The next state relation (22) satisfies (19). Moreover, $\pi_X(\mathcal{B}_S)\,|_{[0,0]} = \{\lambda\}$ as (15) implies $x(0) = w\,|_{[0,-1]} = \lambda$. Hence Σ_ψ is the evolution law induced by Σ_S. As Σ is l-complete (and therefore complete), so is Σ_S, implying that Σ_S is realized by Σ_ψ from Definition 3. □

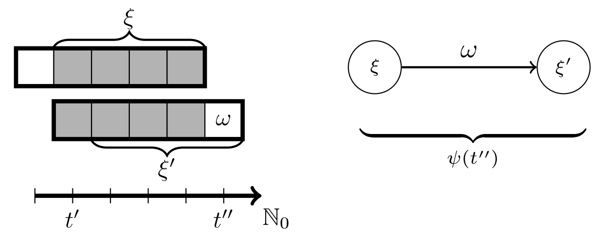

Fig. 2. Correspondence of one step in the domino game (left) to one transition in Σ_ψ (right), where $t' = t'' - l$ and $l = 4$.

Remark 4. Recall the gedankenexperiment in Section 3 and observe that in the construction of Lemma 2 the state represents the "recent past" of the signal w, i.e., a finite string of length l if $t \geq l$. However, at start up, i.e., for $t < l$, no "past" of this length exists. Then the state describes the available past information, i.e., a finite string of length $r \in [0, l-1]$ contained in the set $\mathcal{B}\,|_{[0,r-1]}$. Therefore, assuming $\xi = (\omega_0, \ldots, \omega_{r-1})$ s.t. $|\xi|_L = r < l$ implies that $(\xi, \omega, \xi') \in \psi(t)$ iff $\xi' = \xi \cdot w = (\omega_0, \ldots, \omega_{r-1}, \omega)$ is the extension of ξ by ω and a valid initial behavior, i.e., $\xi' \in \mathcal{B}\,|_{[0,r]}$. Now remember that the domino game describes the admissible behavior by appending domino pieces of length $l+1$ such that the last l symbols match. Therefore, assuming $\xi = (\omega_0, \ldots, \omega_{l-1})$ (i.e., $|\xi|_L = l$) implies $(\xi, \omega, \xi') \in \psi(t)$ iff $\xi' = \xi\,|_{[1,l-1]} \cdot \omega = (\omega_1, \ldots, \omega_{l-1}, \omega)$ and $\xi \cdot \omega = (\omega_0, \ldots, \omega_{l-1}, \omega) \in \mathcal{B}\,|_{[t-l,t]}$, i.e., $\xi \cdot \omega$ is a domino that is currently allowed to be attached. For an illustration of the last case, see Fig. 2.

As every l-complete approximation is l-complete, we have the following immediate consequence of Theorem 3.

Corollary 1. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a dynamical system and $\Sigma^{l\Uparrow} = (\mathbb{N}_0, W, \mathcal{B}^{l\Uparrow})$ its strongest l-complete approximation. Then the state space representation $\Sigma_S^{l\Uparrow} = (\mathbb{N}_0, W, X, \mathcal{B}_S^{l\Uparrow})$ suggested in Lemma 2 can be realized by the evolution law $\Sigma_\psi = (\mathbb{N}_0, W, X, \psi, \{\lambda\})$ with*

$$\begin{aligned} \psi(t) = &\left\{(\xi, \omega, \xi \cdot \omega) | t < l \wedge \xi \cdot \omega \in \mathcal{B}^{l\Uparrow}\,|_{[0,t]}\right\} \\ &\cup \left\{(\xi, \omega, \xi\,|_{[1,l-1]} \cdot \omega) | t \geq l \wedge \xi \cdot \omega \in \mathcal{B}^{l\Uparrow}\,|_{[t-l,t]}\right\}. \end{aligned} \tag{23}$$

Example 4. Using the state spaces derived in Example 3 and the construction of the next state relation in (22), we can construct the evolution laws Σ_{ψ_1} and Σ_{ψ_2}, realizing the strongest 1- and 2-complete approximations of the system Σ in (5), respectively. The evolution laws Σ_{ψ_1} and Σ_{ψ_2} are depicted in Fig. 3, where a transition $(\xi, \omega, \xi') \in \psi_i(t)$ is depicted by an arrow from ξ to ξ' labeled by "ω if $t \in V$", where $V := \left\{t \in \mathbb{N}_0 | (\xi, \omega, \xi') \in \psi(t)\right\}$. Whenever a transition can always occur if its source state ξ is reached, the label reduces to "ω". In both figures the initial state is indicated by an arrow pointing to it from "outside".

As Σ is 2-complete, Σ_{ψ_2} is also a realization of Σ. In Σ_{ψ_2} the transition from state aa to itself is time dependent, as three sequential a's are only allowed at start up. ◁

6. Asynchronous properties

Recall that a major aim of this paper is to construct abstractions representable by an FSM. However, we have seen in the previous section, that the strongest l-complete approximation of a system is generally realized by a time dependent evolution law Σ_ψ. Obviously, one could render the evolution law time independent by using time as an additional state variable. However, this would lead to an infinite state set. We want to characterize systems

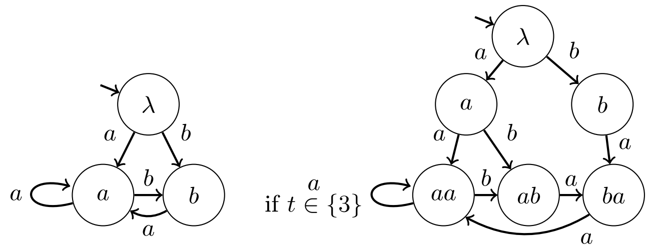

Fig. 3. Evolution laws Σ_{ψ_1} (left) and Σ_{ψ_2} (right) constructed in Example 4, realizing $\Sigma_S^{1\Uparrow}$ and $\Sigma_S^{2\Uparrow}$ of (5), respectively.

naturally allowing an FSM realization. Observe that such systems must allow for concatenation of state trajectories that reach the same state asynchronously (i.e., at different times). This is formalized in the following definition inspired by [19, p. 59].

Definition 4. Let $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ be a dynamical system with internal signal space X. Then Σ_S is an *asynchronous state space dynamical system* if

$$\forall (w_1, x_1), (w_2, x_2) \in \mathcal{B}_S, t_1, t_2 \in \mathbb{N}_0 \,.\, \left(x_1(t_1) = x_2(t_2) \Rightarrow (w_1, x_1) \wedge^{t_1}_{t_2} (w_2, x_2) \in \mathcal{B}_S\right). \tag{24}$$

It can be easily observed that every asynchronous state space dynamical system is also a synchronous[4] state space dynamical system since we can always pick $t_1 = t_2 = t$ in (24) and get (11).

As an important intermediate result, the following proposition shows that the asynchronous state property in (24) implies time independence of the induced evolution law.

Proposition 2. *Let $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ be an asynchronous state space dynamical system. Then the evolution law induced by Σ_S is time independent.*

Proof. Let $\Sigma_\psi = (\mathbb{N}_0, W, X, \psi, X_0)$ be the evolution law induced by Σ_S and $\delta = \bigcup_{t\in\mathbb{N}_0} \psi(t)$. Now observe that (17) and (18) coincide (i.e., $\mathcal{B}_\psi = \mathcal{B}_\delta$) if

$$\forall t \in \mathbb{N}_0 \,.\, \begin{pmatrix} (\xi, \omega, \xi') \in \psi(t) \\ \wedge(\xi, \omega'', \xi'') \in \delta \end{pmatrix} \Rightarrow (\xi, \omega'', \xi'') \in \psi(t), \tag{25}$$

which remains to be shown. Using (19) and the construction of δ, the left hand side of (25) implies

$$\exists (w', x') \in \mathcal{B}_S \,.\, \begin{pmatrix} x'(t) = \xi \\ \wedge w'(t) = \omega \\ \wedge x'(t+1) = \xi' \end{pmatrix} \quad \text{and}$$

$$\exists (w'', x'') \in \mathcal{B}_S, t'' \in \mathbb{N}_0 \,.\, \begin{pmatrix} x''(t'') = \xi \\ \wedge w''(t'') = \omega'' \\ \wedge x''(t''+1) = \xi'' \end{pmatrix}.$$

Now (24) implies $(w, x) = (w', x') \wedge^{t}_{t''} (w'', x'') \in \mathcal{B}_S$ and therefore $x(t) = \xi, w(t) = \omega''$ and $x(t+1) = \xi''$ implying $(\xi, \omega'', \xi'') \in \psi(t)$ (from (19)). □

A simple consequence of Proposition 2, Theorem 2 and Definition 3 is the following corollary.

[4] To clearly distinguish the asynchronous state property and the (standard) state property from Section 4, we will in the remainder of this paper refer to the latter one as *synchronous* state property. The same convention is applied to other properties as memory span and l-completeness.

Corollary 2. *Let $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ be a complete asynchronous state space dynamical system with W and X finite sets, Σ_ψ the evolution law induced by Σ_S and $\delta = \bigcup_{t\in\mathbb{N}_0} \psi(t)$. Then Σ_S can be realized by the FSM $P = (X, W, \delta, \pi_X(\mathcal{B}_S|_{[0,0]}))$.*

Remark 5. It is interesting to note that the asynchronous state property from Definition 4 and the resulting time independence of the corresponding evolution law coincide precisely with the "standard" notion of time invariant state space systems in continuous or discrete time (see, e.g., [21, Ch. 2]). It should be kept in mind though, that this notion does not imply, and is not implied by, the (behavioral) notion of time invariance introduced in Section 2.

Recall that the concepts of synchronous state property and synchronous memory span are strongly related, since the synchronous state property implies that $\Sigma_x = (\mathbb{N}_0, X, \pi_X(\mathcal{B}_S))$ has memory span one. To get the same relation for the asynchronous case, we define an asynchronous memory span.

Definition 5. The dynamical system $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ has *asynchronous memory span l* if

$$\forall w_1, w_2 \in \mathcal{B}, t_1, t_2 \in \mathbb{N}_0 \,.\, \left(w_1|_{[t_1, t_1+l-1]} = w_2|_{[t_2, t_2+l-1]} \Rightarrow w_1 \wedge^{t_1}_{t_2} w_2 \in \mathcal{B}\right). \tag{26}$$

As expected, it can be easily seen that every system with asynchronous memory span l also has synchronous memory span l.

For systems with an asynchronous memory span, the domino game presented in Section 3 is significantly simplified. At any time t we can attach any domino from the whole domino set $\mathcal{D}_{l+1} = \bigcup_{t'\in\mathbb{N}_0} \mathcal{B}|_{[t', t'+l]}$, as long as the first l symbols of the newly attached domino match the last l symbols of the previous domino. Recall that this implies time independent transitions in the induced evolution law, which is what we are aiming at. Having this interpretation in mind, the definition of asynchronous l-completeness comes as no surprise.

Definition 6. The system $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ is *asynchronously l-complete* if

$$\begin{pmatrix} w|_{[0,l]} \in \mathcal{B}|_{[0,l]} \\ \wedge \forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \bigcup_{t'\in\mathbb{N}_0} \mathcal{B}|_{[t',t'+l]} \end{pmatrix} \Leftrightarrow w \in \mathcal{B}. \tag{27}$$

Again, it is easily verified that a system is synchronously l-complete if it is asynchronously l-complete.

Remark 6. The second line in (27) describes that the possible future evolution of the system depends on the l past values of a signal if $t \geq l$. However, at start up this "past" is not yet fully available. Therefore, the first line in (27) is needed to ensure that all signals start with an allowed initial pattern. However, observe that if Σ is time invariant, the condition $\sigma\mathcal{B} \subseteq \mathcal{B}$ implies $\forall t \in \mathbb{N}_0 \,.\, \sigma^t \mathcal{B}|_{[t,t+l]} \subseteq \mathcal{B}|_{[0,l]}$ giving $\bigcup_{t'\in\mathbb{N}_0} \mathcal{B}|_{[t',t'+l]} = \mathcal{B}|_{[0,l]}$. Then the first line in (27) is implied by the second line and is therefore unnecessary. This is stated in the following lemma.

Lemma 3. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a time invariant dynamical system. Then Σ is asynchronously l-complete iff*

$$\left(\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[0,l]}\right) \Leftrightarrow w \in \mathcal{B}. \tag{28}$$

Proof. As pointed out in Remark 6, time invariance of Σ implies $\bigcup_{t'\in\mathbb{N}_0} \mathcal{B}|_{[t',t'+l]} = \mathcal{B}|_{[0,l]}$. Hence (28) and (27) are identical. □

Remark 7. Recall from Remark 1 that in [3] l-completeness for time invariant systems is defined by (28) (instead of (4)). Therefore, Lemma 3 implies that this stronger version of l-completeness from [3] coincides with the property of asynchronous l-completeness for time-invariant systems.

Recall from Proposition 1 that a synchronously l-complete system always has synchronous memory span l, whereas the reverse implication only holds if the system is complete. To emphasize that the asynchronous properties extend the behavioral systems theory in a consistent way, we proof the same correspondence for the asynchronous case.

Proposition 3. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a dynamical system and $l \in \mathbb{N}$. Then*

$$\Sigma \text{ is asynchronously } l\text{-complete} \Leftrightarrow \begin{pmatrix} \Sigma \text{ is complete} \\ \wedge \Sigma \text{ has asynchronous memory span } l \end{pmatrix} \tag{29}$$

Proof. "⇒" As asynchronous l-completeness implies synchronized l-completeness, it also implies completeness. To show that asynchronous l-completeness of Σ implies (26), we fix $w_1, w_2 \in \mathcal{B}, t_1, t_2 \in \mathbb{N}_0$ s.t. $w_1|_{[t_1,t_1+l-1]} = w_2|_{[t_2,t_2+l-1]}$ and show that $w = w_1 \wedge_{t_2}^{t_1} w_2 \in \mathcal{B}$ follows. Observe that for $t_1 = 0$ the statement trivially holds as $w = w_2 \in \mathcal{B}$. For $t_1 > 0$ we have

$$w|_{[t,t+l]} = \begin{cases} w_1|_{[t,t+l]}, & t < t_1 \\ w_2|_{[t-t_1+t_2,t-t_1+t_2+l]}, & t \geq t_1, \end{cases}$$

hence $\forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \bigcup_{t' \in \mathbb{N}_0} \mathcal{B}|_{[t',t'+l]}$. With asynchronous l-completeness of Σ follows $w \in \mathcal{B}$.

"⇐" To show that (27) holds for a complete system Σ with asynchronous memory span l, we fix $w \in W^{\mathbb{N}_0}$ s.t. the left hand side of (27) holds and show that $w \in \mathcal{B}$ follows. Observe that the left hand side of (27) implies

$$\exists w_0, w_1 \in \mathcal{B}, t' \in \mathbb{N}_0 \,.\, \begin{pmatrix} w_0|_{[0,l]} = w|_{[0,l]} \\ \wedge w_1|_{[t',t'+l]} = w|_{[1,1+l]} \\ \wedge w_0|_{[1,l]} = w_1|_{[t',t'+l-1]} \end{pmatrix}. \tag{30}$$

As Σ has asynchronous memory span l, the last line in (30) implies $w_0 \wedge_{t'}^{1} w_1 \in \mathcal{B}$ (from (26)), hence $\left(w_0 \wedge_{t'}^{1} w_1\right)|_{[0,l+1]} = w|_{[0,l+1]} \in \mathcal{B}|_{[0,l+1]}$. Iteratively applying this procedure therefore yields $\forall \tau \in \mathbb{N}_0 \,.\, w|_{[0,\tau]} \in \mathcal{B}|_{[0,\tau]}$ implying $w \in \mathcal{B}$ as Σ is complete. □

Example 5. We now investigate the asynchronous l-completeness properties of the system Σ in (5). Since Σ is time invariant, it follows from Remark 6 that $\bigcup_{t' \in \mathbb{N}_0} \mathcal{B}|_{[t',t'+l]} = \mathcal{B}|_{[0,l]}$. Therefore, the simplified domino game for $l = 1$ is identical to the one played in Example 1, implying that the system (5) is not asynchronously 1-complete. For $l = 2$, observe that in the simplified domino game we are still allowed to use the piece *aaa* from the set $\mathcal{B}|_{[0,2]}$ at any time $t > 0$. Therefore, more than two sequential a's can be produced by this game implying that the system (5) is *not* asynchronously 2-complete. Extending l to $l = 3$ gives the domino set $\bigcup_{t' \in \mathbb{N}_0} \mathcal{B}|_{[t',t'+3]} = \mathcal{B}|_{[0,3]} = \{aaab, aaba, abaa, baab\}$. Now, playing the simplified domino game ensures that always three symbols have to match, preventing the piece *aaab* to be attachable for $t > 0$. Hence, the resulting behavior is identical to $\mathcal{B}$. This implies that the system (5) is asynchronously 3-complete. ◁

If we recall that the memory of the system is still given by the last l symbols of the signal w it is obvious that we can construct a state space representation of an asynchronously l-complete system exactly as given in Lemma 2 for a system with synchronous memory span l. However, dealing with the asynchronous version, we can realize it by an FSM.

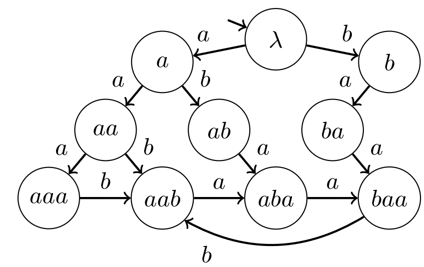

Fig. 4. FSM P realizing (5).

Theorem 4. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be an asynchronously l-complete dynamical system. Then $\Sigma_S = (\mathbb{N}_0, W, X, \mathcal{B}_S)$ from Lemma 2 is an asynchronous state space representation of Σ. Furthermore, if $|W| < \infty$, Σ_S is realized by the finite state machine $P = (X, W, \delta, X_0)$ with $X_0 = \{\lambda\}$ and*

$$\delta = \left\{(\xi, \omega, \xi \cdot \omega) \,|\, |\xi|_L < l \wedge \xi \cdot \omega \in \mathcal{B}|_{[0,|\xi|_L]}\right\}$$
$$\cup \left\{(\xi, \omega, \xi|_{[1,l-1]} \cdot \omega) \,|\, |\xi|_L = l \wedge \xi \cdot \omega \in \bigcup_{t' \in \mathbb{N}_0} \mathcal{B}|_{[t',t'+l]}\right\}. \tag{31}$$

Proof. The proof of the first part follows the same lines as the proof of Lemma 2 and is therefore omitted. To prove the second part, observe that $\delta = \bigcup_{t \in \mathbb{N}_0} \psi(t)$ with $\psi(t)$ from (22). It was furthermore shown in the proof of Theorem 3 that Σ_ψ is the evolution law induced by Σ_S. Moreover, l-completeness of Σ implies completeness of Σ_S. As finiteness of W implies finiteness of X from (15), it follows from Corollary 2 that Σ_S is realized by the finite state machine $P = (X, W, \delta, X_0)$. □

Remark 8. The next state relation δ in (31) can be interpreted analogously to ψ in (22), see the discussion in Remark 4. Observe that now the condition in the last line of (31) is weakened in the sense that $\xi \cdot \omega$ can be any domino in the gedankenexperiment in Section 3.

Example 6. Recall from Example 5 that the system (5) in Example 1 is asynchronously 3-complete and that (7) implies $\bigcup_{t \in \mathbb{N}_0} \mathcal{B}|_{[t,t+2]} = \{aaa, aab, aba, baa\}$. Adding the set $\bigcup_{r \in [0,2]} \mathcal{B}|_{[0,r-1]} = \{\lambda, a, b, aa, ab, ba\}$, the state space defined in Lemma 2 with $l = 3$ for the system in (5) is given by $X = \{\lambda, a, b, aa, ab, ba, aaa, aab, aba, baa\}$. Using this state space and the construction of the next state relation in (31), we can construct an FSM P realizing the system (5) in Example 1. The result is depicted in Fig. 4. ◁

Remark 9. Observe that (8) in Lemma 1 and (28) in Lemma 3 are identical. Therefore, Lemmas 1 and 3 imply that the asynchronous and the synchronous l-completeness property coincide for strictly time invariant systems. As a direct consequence, the state space representation of a strictly time invariant (synchronously) l-complete system can be realized by the FSM P constructed in Theorem 4.

7. Asynchronous *l*-complete approximation

Using the asynchronous l-completeness property introduced in Definition 6, we can construct asynchronous l-complete approximations analogously to their synchronous versions in Section 3.

Definition 7. Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a dynamical system. Then $\Sigma^l = (\mathbb{N}_0, W, \mathcal{B}^l)$ is an *asynchronous l-complete approximation* of Σ, if (i) Σ^l is asynchronously l-complete and (ii) $\mathcal{B}^l \supseteq \mathcal{B}$. Furthermore, $\Sigma^{l\uparrow} = (\mathbb{N}_0, W, \mathcal{B}^{l\uparrow})$ is the *strongest asynchronous l-complete approximation* of Σ, if (i) $\Sigma^{l\uparrow}$ is an asynchronous l-complete approximation of Σ and (ii) for any asynchronous l-complete approximation $\Sigma' = (\mathbb{N}_0, W, \mathcal{B}')$ of Σ it holds that $\mathcal{B}^{l\uparrow} \subseteq \mathcal{B}'$.

Recall that for an asynchronously l-complete system, the domino game gedankenexperiment can be simplified such that at any time t we can attach any domino from the whole domino set $\mathcal{D}_{l+1} = \bigcup_{t'\in\mathbb{N}_0} \mathcal{B}|_{[t',t'+l]}$. This simplified domino game now constructs the unique strongest asynchronous l-complete approximation $\mathcal{B}^{l\uparrow}$.

Theorem 5. *Let $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ be a dynamical system. Then the unique strongest asynchronous l-complete approximation of Σ is given by $\Sigma^{l\uparrow} = (\mathbb{N}_0, W, \mathcal{B}^{l\uparrow})$, with*

$$\mathcal{B}^{l\uparrow} := \left\{ w \middle| \begin{pmatrix} w|_{[0,l]} \in \mathcal{B}|_{[0,l]} \\ \wedge \forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \bigcup_{t'\in\mathbb{N}_0} \mathcal{B}|_{[t',t'+l]} \end{pmatrix} \right\}. \tag{32}$$

Furthermore, if Σ is time invariant then

$$\mathcal{B}^{l\uparrow} = \left\{ w \in W^{\mathbb{N}_0} | \forall t \in \mathbb{N}_0 \,.\, w|_{[t,t+l]} \in \mathcal{B}|_{[0,l]} \right\}. \tag{33}$$

Proof. The proof of the first part follows the same lines as the proof of Theorem 1, and the second part follows directly from Lemma 3. □

The following corollary is a direct consequence of Theorems 4 and 5.

Corollary 3. *The strongest asynchronous l-complete approximation $\Sigma^{l\uparrow} = (\mathbb{N}_0, W, \mathcal{B}^{l\uparrow})$ of any dynamical system $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ with finite external signal space W can be represented by the asynchronous state space system $\Sigma_S^{l\uparrow} = (\mathbb{N}_0, W, X, \mathcal{B}_S^{l\uparrow})$ with X and $\mathcal{B}_S^{l\uparrow}$ as in Lemma 2. $\Sigma_S^{l\uparrow}$ can be realized by the FSM $P = (X, W, \delta, X_0)$ given in Theorem 4.*

Remark 10. The construction of the FSM P realizing $\Sigma^{l\uparrow}$ relies on the computation of the set $\bigcup_{t'\in\mathbb{N}_0} \mathcal{B}|_{[t',t'+l]}$, which is in general not a trivial task. However, a finite external signal space W implies the existence of a finite time $t_{ti} \in \mathbb{N}_0$ s.t. Σ becomes time invariant for $t > t_{ti}$. For this case, [17, p. 51] proposes a pragmatically motivated extension of the strongest l-complete approximation in the sense of [3]. It can be easily observed that the behavior $\mathcal{B}_l^\star$ constructed in [17, p. 51] coincides with $\mathcal{B}^{l\uparrow}$ in (32). As the approach in [17, p. 51] is pragmatically motivated, this coincidence underlines the relevance of our theoretically developed notion.

Remark 11. As a direct consequence of Remark 9, the strongest synchronous and the strongest asynchronous l-complete approximation coincide for strictly time invariant systems. This can be easily seen by noting that (10) in Theorem 1 and (33) in Theorem 5 coincide.

Furthermore, recall from Remark 7 that the stronger notion of l-completeness from [3] coincides with the property of asynchronous l-completeness for time-invariant systems. Therefore, the strongest l-complete approximation of a time invariant system Σ suggested in [3] is identical to its strongest asynchronous l-complete approximation $\Sigma^{l\uparrow}$ introduced in Definition 7. The latter is, by definition, also a synchronous l-complete approximation, but not necessarily the strongest one.

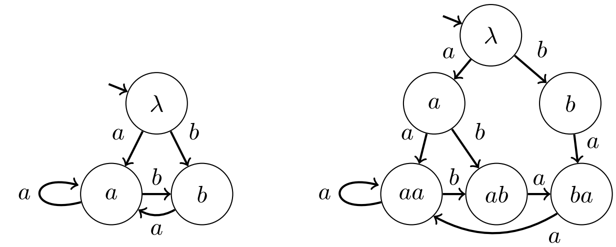

Fig. 5. FSMs realizing the strongest asynchronous 1-complete approximation $\Sigma_S^{1\uparrow}$ (left) and the strongest asynchronous 2-complete approximation $\Sigma_S^{2\uparrow}$ (right) of (5).

Example 7. The behaviors constructed by the domino games discussed in Example 5 characterize the strongest asynchronous 1-, 2- and 3-complete approximations for the system Σ in (5), respectively. Realizations for the strongest asynchronous 1- and 2-complete approximations using the constructions from Theorem 4 are shown in Fig. 5. As the system Σ is asynchronously 3-complete, its behavior coincides with that of its strongest asynchronous 3-complete approximation; hence the corresponding FSM is shown in Fig. 4. Observe that the FSM realizing $\Sigma_S^{1\uparrow}$ and the evolution law Σ_{ψ_1} realizing $\Sigma_S^{1\Uparrow}$ depicted in Fig. 5 (left) and Fig. 3 (left), respectively, coincide. This is a direct consequence from Remark 9, since $\Sigma^{1\Uparrow}$ is strictly time invariant as discussed in Example 2. ◁

Summarizing the results of our running example, we have the following: the system under consideration, $\Sigma = (\mathbb{N}_0, W, \mathcal{B})$ in (5), is time invariant but not strictly time invariant. Σ is synchronously 2-complete and can therefore be realized by the time dependent evolution law Σ_{ψ_2} depicted in Fig. 3 (right). It is asynchronously 3-complete (but not asynchronously 2-complete) and can therefore be realized by an FSM P depicted in Fig. 4. Its strongest asynchronous 2-complete approximation $\Sigma^{2\uparrow}$ is of cause also a synchronous 2-complete approximation of Σ, but not the strongest one. In fact, as Σ is synchronously 2-complete and asynchronously 3-complete, $\Sigma^{2\Uparrow} = \Sigma^{3\uparrow} = \Sigma$, and therefore $\mathcal{B}^{2\Uparrow} = \mathcal{B}^{3\uparrow} = \mathcal{B} \subset \mathcal{B}^{2\uparrow}$.

As the system (5) is asynchronous 3-complete, it can of course be realized by an FSM. From an application point of view, (asynchronous) l-complete approximations are of course much more important, if the system to be approximated does not have an FSM realization. This scenario is illustrated by the following simple example.

Example 8. Consider the system

$$\Sigma = (\mathbb{N}_0, W, \mathcal{B}) \quad s.t. \tag{34}$$
$$\mathcal{B} = \{a^n b^n c^\omega \mid n \in \mathbb{N}, n \geq m\},$$

where $m \in \mathbb{N}$ is a fixed number and $(\cdot)^\omega$ denotes the infinite repetition of the respective string. Observe that Σ is time variant and $\mathcal{B}$ is the limit of the deterministic context free language $L = \{a^n b^n c^* \mid n \in \mathbb{N}, n \geq m\}$, not allowing for an FSM realization (see, e.g., [22]).

The evolution law Σ_ψ realizing the strongest synchronous 1-complete approximation $\Sigma_S^{1\Uparrow}$ of Σ is depicted in Fig. 6 (top), inducing the behavior

$$\mathcal{B}_\psi = \mathcal{B}^{1\Uparrow} = \{a^m a^* b^m b^* c^\omega\} \supset \mathcal{B}.$$

The FSM P realizing the strongest asynchronous 1-complete approximation $\Sigma_S^{1\uparrow}$ of Σ is depicted in Fig. 6 (bottom), inducing the behavior

$$\mathcal{B}_\delta = \mathcal{B}^{1\uparrow} = \{aa^* bb^* c^\omega\} \supseteq \mathcal{B}^{1\Uparrow} \supset \mathcal{B}.$$

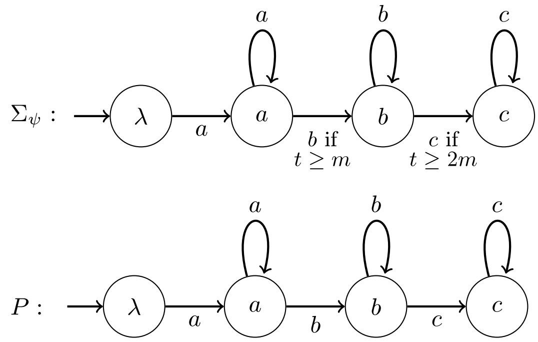

Fig. 6. Evolution law Σ_ψ (top) and FSM P (bottom) realizing the strongest synchronous and the strongest asynchronous 1-complete approximation of (34), respectively.

Furthermore, the behavior of the strongest asynchronous m-complete approximation of Σ is given by

$$\mathcal{B}^{m^{\Uparrow}} = \{a^m a^* b^m b^* c^\omega\} = \mathcal{B}^{1^{\Uparrow}} \supset \mathcal{B},$$

generating the same behavior as the strongest synchronous 1-complete approximation. However, $\mathcal{B}^{m^{\Uparrow}}$ can be realized using an FSM. ◁

8. Conclusion

Strongest l-complete approximations for time invariant systems were introduced in [3]. However, the employed notion of l-completeness is a stronger version of the original l-completeness property defined in [14]. To resolve the resulting inconsistencies, and also to address a wider system class, the procedure suggested in [3] can be adapted in a straightforward way using the original l-completeness notion from [14], capturing also time variant systems. This, not surprisingly, leads to realizations with time dependent next state relations. To address this, inspired by [18], we have extended the well-known concepts of state property, memory span and l-completeness and have introduced asynchronous versions of these concepts. To clearly distinguish between the new, stronger versions and the original ones, the latter ones are referred to as synchronous properties.

Based on these extensions, we have proposed a new approximation technique, called strongest asynchronous l-complete approximation. For systems with finite external signal space, it generates a finite state machine (FSM) as realization of the approximation. For time invariant systems, it produces the same approximation as [3], however, the mentioned inconsistencies are resolved. The strongest asynchronous l-complete approximation of a given system is also a synchronous l-complete approximation, but not necessarily the strongest one. For strictly time invariant systems, we have shown that the concepts of strongest synchronous and strongest asynchronous l-complete approximations coincide.

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