

On the left eigenstructure assignment and state feedback design

Yashar Kouhi and Naim Bajcinca

Abstract—A general method for assigning the left eigenvectors and their corresponding eigenvalues for the single- and multi-input LTI systems using linear state feedback is proposed. Moreover, the concept of the common left eigenstructure assignment is shown to be useful for the exponential stabilization of switching linear systems under arbitrary switching conditions.

I. LEFT EIGENSTRUCTURE ASSIGNMENT

A. Single-input systems

Consider the state space representation of a single input LTI system

$$\dot{x} = Ax + bu, \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the state and the control input, respectively. Suppose we want to construct a state feedback $u = k^T x$, which is to assign a given left eigenvector $w \in \mathbb{R}^n$, and a corresponding eigenvalue $\lambda_0 \in \mathbb{R}_{<0}$ to the closed loop system $A_{cl} = A + bk^T$. Then, by simple computation

$$w^T(A + bk^T) = \lambda_0 w^T \implies k^T = -\frac{w^T(A - \lambda_0 I)}{b^T w}, \quad (2)$$

where $b^T w \neq 0$ is assumed, and I stands for the unity matrix of the order n .

A natural problem consists now in exploring the proper selection of w and λ_0 such that the closed loop system matrix $A_{cl} = A + bk^T$ given by

$$A_{cl} = \left(I - \frac{bw^T}{b^T w} \right) A + \lambda_0 \frac{bw^T}{b^T w}. \quad (3)$$

is Hurwitz. To this end, consider first the following statement.

Proposition 1: The characteristic polynomial of the closed loop system A_{cl} is given by

$$\begin{aligned} |\lambda I - A_{cl}| &= (\lambda - \lambda_0) \frac{b^T \text{adj}(\lambda I - A^T) w}{b^T w} \\ &= (\lambda - \lambda_0) (\lambda^{n-1} + \beta_1 \lambda^{n-2} + \dots + \beta_{n-1}), \end{aligned} \quad (4)$$

where

$$\beta_i = \frac{b^T (a_i I + a_{i-1} A + \dots + A^i)^T w}{b^T w}, \quad (5)$$

($i=1, \dots, n-1$) and a_1, \dots, a_n stand for the coefficients of the characteristic polynomial of A

$$p(\lambda) = |\lambda I - A| = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n. \quad (6)$$

Y. Kouhi and N. Bajcinca are with *Max Planck Institute for Dynamics of Complex Technical Systems*, Sandtorstr. 1, 39106 Magdeburg, Germany. Corresponding email: bajcinca@mpi-magdeburg.mpg.de.

Proof: The characteristic polynomial of the closed loop system reads

$$\begin{aligned} |\lambda I - A - bk^T| &= \\ &= |\lambda I - A| |I - (\lambda I - A)^{-1} bk^T| \\ &= p(\lambda) (1 - k^T (\lambda I - A)^{-1} b) \\ &= p(\lambda) \left(1 + \frac{w^T (A - \lambda_0 I) (\lambda I - A)^{-1} b}{b^T w} \right) \\ &= p(\lambda) \left(\frac{w^T (\lambda I - A - \lambda_0 I + A) (\lambda I - A)^{-1} b}{b^T w} \right) \\ &= (\lambda - \lambda_0) \frac{b^T \text{adj}(\lambda I - A^T) w}{b^T w}. \end{aligned}$$

[*Hint:* For the above derivation steps the reader is referred to Section 3.2 in [1].] Equation (5) results now directly after comparing the coefficients in the polynomial identity

$$\begin{aligned} \lambda^{n-1} + \alpha_1 \lambda^{n-1} + \dots + \alpha_n &= \\ &= (\lambda - \lambda_0) (\lambda^{n-1} + \beta_1 \lambda^{n-2} + \dots + \beta_{n-1}), \end{aligned}$$

where the coefficients $\alpha_1, \dots, \alpha_n$ are given by the Bass-Gura formula, see [1]

$$\begin{aligned} \alpha_1 &= a_1 - k^T b, \\ \alpha_2 &= a_2 - k^T A b - a_1 k^T b, \\ \alpha_3 &= a_3 - k^T A^2 b - a_1 k^T A b - a_2 k^T b, \end{aligned}$$

and so forth. ■

Example 1: For $n = 2$, it follows from (5) that for the stability one needs

$$\beta_1 = \frac{b^T (a_1 I + A)^T w}{b^T w} > 0, \quad (7)$$

which is satisfied if the inner-products of w with b and $(a_1 I + A)b$ share the same sign.

Due to the nonlinear appearance of the eigenvector w in (5), applying the Routh-Hurwitz stability criteria for the construction of w is, in general, tedious. In the following we provide a systematic method for the selection of all stabilizing left eigenvector w in the general case. For the sake of simplicity, assume in the first step that the system in (1) is given in the canonical controller form

$$\dot{x} = A_c x + b_c u \quad (8)$$

with

$$A_c = \begin{pmatrix} 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{pmatrix}, \quad b_c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Without loss of generality assume further that the left eigenvector is given as $w_c = [w_{c,1} \dots w_{c,n-1} 1]^T$, where $w_{c,n} = 1$ is set to assure $b_c^T w_c = 1$. It is an easy exercise to show that

$$b_c^T \text{adj}(\lambda I - A_c^T) = [1 \ \lambda \ \dots \ \lambda^{n-1}]. \quad (9)$$

Then, with the feedback control $u = k_c^T x$ with $k_c^T = -w_c^T (A_c - \lambda_0 I)$, and referring to (4), it follows that

$$|\lambda I - A_{c,\text{cl}}| = (\lambda - \lambda_0)(\lambda^{n-1} + w_{c,n-1} \lambda^{n-2} + \dots + w_{c,1}). \quad (10)$$

Hence, the closed loop characteristic polynomial is independent of the parameters a_1, \dots, a_n of the open loop matrix A_c . Moreover, from (3), the closed loop matrix $A_{c,\text{cl}} = A_c + b_c k_c^T$ reads

$$A_{c,\text{cl}} = \left(I - \frac{b_c w_c^T}{b_c^T w_c} \right) A_c + \lambda_0 \frac{b_c w_c^T}{b_c^T w_c}. \quad (11)$$

[Note: For convenience, with regard to the comparison with (12), we keep here the term $b_c^T w_c = 1$ in the denominator.]

Each controllable system (1) can be converted into the controller canonical form using the transformation $T = \Phi_c \Phi_{cc}^{-1}$, where Φ_c and Φ_{cc} are the controllability matrices of the original and of the transformed system, respectively. Then, $A_c = T^{-1} A T$ and $b_c = T^{-1} b$ applies, and we get

$$\begin{aligned} T^{-1} A_{\text{cl}} T &= T^{-1} A T - \frac{(T^{-1} b) w^T A T}{b^T w} + \lambda_0 \frac{(T^{-1} b) w^T T}{b^T w} \\ &= \left(I - \frac{b_c (w^T T)}{b_c^T (T^T w)} \right) A_c + \lambda_0 \frac{b_c (w^T T)}{b_c^T (T^T w)}. \end{aligned} \quad (12)$$

A comparison of this expression with (11), implies that the condition $T^{-1} A_{\text{cl}} T = A_{c,\text{cl}}$ holds if $w^T = \gamma \cdot w_c^T T^{-1}$, where γ is any nonzero real scalar. This closes the proof of the following statement.

Theorem 1: Let the single-input LTI system (1) be controllable, and define $T = \Phi_c \Phi_{cc}^{-1}$. Assume $\lambda_0 < 0$, and consider any Hurwitz polynomial in the form (10). Then, the closed loop matrix (3) is Hurwitz, if the left eigenvector w is given by $w = \gamma \cdot (T^{-1})^T w_c$, for some real nonzero γ , and $w_c = [w_{c,1} \dots w_{c,n-1} 1]^T$.

B. Multi-input systems

In this section, we provide the generalized results for the state space representation of a multi-input system

$$\dot{x} = Ax + Bu, \quad (13)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are again the state and the control input. Let $w_i \in \mathbb{R}^n$, $i = 1, \dots, m$ be m given linearly independent vectors, and let $\lambda_i \in \mathbb{R}_{<0}$. Define $W^T = [w_1 \dots w_m]$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$. It turns out that the left eigenstructure problem, consisting in the computation of the state feedback $u = K^T x$, such that the closed loop matrix $A_{\text{cl}} = A + BK^T$ are assigned the left eigenvectors in W , with the corresponding eigenvalues in Λ , and $W^T B$ assumed to be nonsingular, is solved by

$$K^T = -(W^T B)^{-1} (W^T A - \Lambda W^T), \quad (14)$$

leading to the closed loop system matrix

$$A_{\text{cl}} = (I - B(W^T B)^{-1} W^T) A - B(W^T B)^{-1} \Lambda W^T. \quad (15)$$

Using $T = [W^T \ W_1^T]^T$, where $W_1^T \in \mathbb{R}^{(n-m) \times n}$ is an arbitrary matrix such that T is nonsingular, it can be shown that, in addition to the eigenvalues $\lambda_1, \dots, \lambda_m$ [also representing the m nonzero eigenvalues of the second summand in (15)], the matrix $T A_{\text{cl}} T^{-1}$, and therefore the matrix A_{cl} , too, possesses $n-m$ eigenvalues equal to the nonzero eigenvalues of the first summand $(I - B(W^T B)^{-1} W^T) A$. Hence, to stabilize (13) by left eigenstructure assignment using (14), the matrix of left eigenvectors W should be appropriately designed such that the underlying $n-m$ eigenvalues lie in the open left half of the complex plane.

II. SWITCHING SYSTEMS

Consider a switching linear system (see [3]) defined as

$$\dot{x} = A_\sigma x(t), \quad (16)$$

where $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \dots, m\}$ is a piecewise continuous function referred to as the switching signal between the constituent systems (or modes) $A_i \in \mathcal{A} = \{A_1, \dots, A_m\}$, representing a collection of matrices in $\mathbb{R}^{2 \times 2}$. Let all matrices $A_i \in \mathcal{A}$ share a same left eigenvector w , with some corresponding eigenvalues $\lambda_{1i} < 0$, $i \in \{1, \dots, m\}$. Then, they must share a common right eigenvector v , too, whose eigenvalues are, say, $\{\lambda_{2i}\}$, and $w^T v = 0$ holds. Our claim is that the exponential stability of the switching system (16) is guaranteed by the stability of each its constituent subsystem $A_i \in \mathcal{A}$.

To prove this statement, we consider the common quadratic Lyapunov function $V(x) = x^T P x$, where $P = w w^T + \epsilon^2 v v^T$. Therefore, we need to show that an $\epsilon^2 > 0$ exists, such that P satisfies the Lyapunov inequality for each $A_i \in \mathcal{A}$, that is

$$A_i^T P + P A_i = \begin{pmatrix} \lambda_{1i} & \frac{1}{2} \epsilon^2 v^T A_i w \\ \frac{1}{2} \epsilon^2 v^T A_i w & \epsilon^2 \lambda_{2i} \end{pmatrix} < 0. \quad (17)$$

Indeed, it can be shown that for a sufficiently small $\epsilon^2 > 0$ in accordance with

$$\epsilon^2 < \min \left\{ \frac{4|A_i|}{(v^T A_i w)^2}, \quad i = 1, \dots, m \right\}. \quad (18)$$

all matrices $A_i^T P + P A_i$ become strictly negative definite.

It should be now obvious that an open loop switching system

$$\dot{x} = A_\sigma x(t) + b_\sigma u, \quad (19)$$

with b_σ taking the values in a given set $\{b_1, \dots, b_m\}$, is exponentially stabilizable by the state feedback controller $u = k_\sigma x$, where $k_\sigma \in \{k_1, \dots, k_m\}$, if a common left eigenvector w exists, such that the closed loop matrices $A_{\text{cl},i} = A_i + b_i k_i^T$ for all $i \in \{1, \dots, m\}$ are Hurwitz, with k_i designed in accordance with the formula (2) and (7).

REFERENCES

- [1] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ; 1980.
- [2] G. P. Liu, R.J. Patton, *Eigenstructure Assignment for Control System Design*, John Wiley and Sons, 1998.
- [3] R. Shorten, F. Wirth, O. Mason, K. Wulff, C. King, "Stability criteria for switched and hybrid systems" *SIAM Rev.*, Vol.49, No.4, pp. 545–592, 2007.