Consistent Control Hierarchies with Top Layers Represented by Timed Event Graphs

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***Abstract*—To handle complexity in large scale control problems, a popular approach is to impose hierarchical control structures. Hierarchical control can be interpreted as an attempt to handle complex problems by decomposing them into smaller subproblems and reassembling their solutions into a "functioning" hierarchical structure. It typically involves a number of control layers operating on different time scales that may be clock driven or event driven. Signals on different levels of the control hierarchy may be of different granularity representing phenomena like measurement aggregation when passing from lower to higher level control. Within a suitable formal framework guaranteeing certain consistency conditions, complexity reduction is achieved by interpreting specifications for lower control levels as abstractions of the plant under low level control. An essential task within a hierarchical control synthesis procedure is then to come up with a suitable choice of specifications for the individual control layers. Because of the dual role of these specifications, this typically involves a non-trivial trade-off. E.g., imposing a less strict specification for a control layer will facilitate the control synthesis task for this layer, but will make the control synthesis task for higher level control more difficult. In this paper, this trade-off is formally investigated for a specific scenario, where the top control layer is only responsible for the timing of certain discrete events, and where the abstraction it is based on can be represented by a timed event graph (TEG).**

I. Introduction

To handle complexity in large scale control problems, a popular approach is to impose hierarchical control structures. To guarantee that the interaction of different control layers does indeed enforce the required set of specifications, a formal framework providing certain consistency conditions is needed. A specific framework for hierarchical control providing such conditions was described in [1], [2], and is set within Willems' behavioural systems theory (e.g.,[3], [4]). It allows for the fact that signals on different levels of the control hierarchy may be of different granularity representing phenomena like measurement aggregation when passing from lower to higher level control. The usual set-up within this scenario is that the states of low-level components "live" in Euclidean state spaces, whereas the top level controller can be represented by a finite automaton. The latter's task is to take logical decisions upon observing the occurrence of certain discrete events, and its synthesis is based on a finite state approximation of the plant under low-level control. In this paper, we investigate a different set-up: it is characterised by the fact that the task to be accomplished by the top-level controller is the timing, but not the ordering, of input events. However, it employs the same abstraction-based philosophy as [2], where a considerable reduction of complexity follows from the fact that the specification for each control layer can be interpreted as an abstraction of the plant under low-level control, and this abstraction is subsequently used for the synthesis of the next (from bottom to top) control layer. Obviously, the set of control layer specifications, when taken together, must be at least as strict as the given total specification for the overall control system. Moreover, the dual role of individual control layer specifications implies a non-trivial trade-off when defining a suitable set of specifications. For example, if a less strict specification for a control layer is imposed, this will facilitate the control synthesis task for this layer, but will make the control synthesis task for higher level control more difficult: as the specification also serves as an abstraction on which synthesis for the next (from bottom to top) control layer is based, this abstraction will provide a less accurate picture of the plant under low-level control. In this paper, we will formally investigate this trade-off for the specific scenario described above: we assume that the top-level controller's sole task is to decide about the timing of certain input events and that all other (discrete and continuous) control inputs are provided by lower control layers. The synthesis of the top-layer controller can then be based on a Timed Event Graph (TEG) abstraction of the plant under low-level control. A TEG is a timed Petri net where each place has exactly one input and one output transition. Such Petri nets can model synchronisation of events but not conflicts with respect to the ordering of events. They are therefore particularly suited for our scenario, where the ordering of events does not belong to the top-layer control tasks.

It is a well known fact that the temporal evolution of

timed event graphs becomes linear in certain tropical algebras. Tropical algebras, or dioid algebras, are idempotent semirings, and we specifically consider the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ (e.g., [5]). It can be interpreted as a combination of the well known $(\min,+)$ and $(\max,+)$ algebras; formally, it is defined as a quotient dioid on the set of two-dimensional power series with Boolean coefficients and integer exponents. As there exists a well-developed control theory for systems that are linear in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ (e.g., [6]), synthesis of the top-level controller is straightforward, if an adequate TEG abstraction of the plant under low-level control is available.

This paper is organised as follows: Section II summarises the relevant algebraic preliminaries by collecting a number of important facts on dioid and residuation theory. Section III explains how TEGs can be conveniently modelled using the specific dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Section IV explores the indicated trade-off in the considered hierarchical setting, and Section V provides an illustrating example.

II. Algebraic Preliminaries

A. Dioid Theory

The following is a short summary of basic results from dioid theory. The reader is invited to consult ([5]) for more details.

Definition 1 (Dioid, Complete Dioid): A dioid $\mathcal{D}$ is a set endowed with two internal operations denoted $\oplus$ (addition) and $\otimes$ (multiplication, often denoted by juxtaposition), both associative and both having a neutral element denoted ε and e respectively. Moreover, $\oplus$ is commutative and idempotent ($\forall a \in \mathcal{D}, a \oplus a = a$), $\otimes$ is distributive with respect to $\oplus$, and ε is absorbing for $\otimes$ ($\forall a \in \mathcal{D}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$).

A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. The sum of all its elements is denoted $\top$.

Definition 2 (Order relation): A dioid is endowed with a partial order denoted $\succeq$ and defined by the following equivalence: $a \succeq b \Leftrightarrow a = a \oplus b$. $a \oplus b$ is the least upper bound of $\{a, b\}$.

Remark 1: A well-known complete dioid is the $(\max,+)$-algebra (resp. $(\min,+)$-algebra): $\mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with $\max$ (resp. $\min$) as addition and $+$ as multiplication. ε is equal to $-\infty$ (resp. $+\infty$) and $\top$ to $+\infty$ (resp. $-\infty$). The associated order relation is equal (resp. dual) to the classical order relation on $\mathbb{Z}$.

Remark 2: Consider a complete dioid $\mathcal{D}$, for all $A, B \in \mathcal{D}^{n\times p}$ and $C \in \mathcal{D}^{p\times m}$, the following operations are defined:

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \tag{1}$$

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{p} A_{ik} C_{kj} \tag{2}$$

Proposition 1 ([7]): Consider a complete dioid $\mathcal{D}$, $\mathcal{D}^{n\times n}$ endowed with the previous operations is a complete dioid.

Definition 3 (Interval): An interval in dioid $\mathcal{D}$ is a set of the form $\mathbf{a} = [\underline{a}, \overline{a}] = \{t \in \mathcal{D} | \underline{a} \preceq t \preceq \overline{a}\}$ with $\underline{a} \preceq \overline{a}$. $\underline{a}$ (resp. $\overline{a}$) is the lower (resp. upper) bound of $\mathbf{a}$. The set of all intervals in a dioid $\mathcal{D}$ is denoted $I(\mathcal{D})$.

Remark 3 ([8]): Consider a complete dioid $\mathcal{D}$, for all $\mathbf{a}, \mathbf{c} \in I(\mathcal{D})$, the following operations are defined on $I(\mathcal{D})$:

$$\mathbf{a} \oplus \mathbf{c} = [\underline{a} \oplus \underline{c}, \overline{a} \oplus \overline{c}] \tag{3}$$

$$\mathbf{a} \otimes \mathbf{c} = [\underline{a} \otimes \underline{c}, \overline{a} \otimes \overline{c}] \tag{4}$$

Proposition 2 ([9]): Consider a complete dioid $\mathcal{D}$, $I(\mathcal{D})$ endowed with the previous operations is a complete dioid.

Theorem 1 (Kleene star theorem): The implicit equation $x = ax \oplus b$ defined over a complete dioid admits $x = a^* b$ as least solution with $a^* = \bigoplus_{i\geqslant 0} a^i$ (Kleene star).

Remark 4: Consider a complete dioid $\mathcal{D}$, for all $a, b \in \mathcal{D}$, $a \succeq b$ implies $a^* b^* = a^*$ and $b^* a^* = a^*$.

B. Residuation Theory

In ordered sets, like dioids, equations $f(x) = b$ may have either no solution, one solution, or multiple solutions. In order to give always a unique answer to the problem of mapping inversion, residuation theory ([10]) provides, under some assumptions, either the greatest solution (in accordance with the considered order) to the inequality $f(x) \preceq b$ or the least solution to the inequality $f(x) \succeq b$.

Definition 4 (Isotone mapping, antitone mapping): A mapping f defined over ordered sets is said to be isotone (resp. antitone) if $a \preceq b \Rightarrow f(a) \preceq f(b)$ (resp. $f(a) \succeq f(b)$).

Definition 5 (Residuation): Let $f : \mathcal{E} \to \mathcal{F}$, with $(\mathcal{E}, \preceq)$ and $(\mathcal{F}, \preceq)$ ordered sets. An isotone mapping f is said to be residuated if for all $y \in \mathcal{F}$, the least upper bound of the subset $\{x \in \mathcal{E} | f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^\sharp(y)$, and mapping $f^\sharp$ is called the residual of f.

Theorem 2 ([5]): Let $f : \mathcal{D} \to \mathcal{E}$ be an isotone mapping defined over complete dioids. Mapping f is residuated if and only if $f(\varepsilon) = \varepsilon$ and, $\forall A \subseteq \mathcal{D}$, $f(\bigoplus_{x\in A} x) = \bigoplus_{x\in A} f(x)$.

Corollary 1: Let $L_a : x \mapsto a \otimes x$ and $R_a : x \mapsto x \otimes a$ be defined on a complete dioid. Mappings L_a and R_a are both residuated. Their residuals will be denoted respectively $L_a^\sharp(x) = a \backslash\!\!\diamond x$ (left-division) and $R_a^\sharp(x) = x \diamond\!\!/ a$ (right-division).

Proposition 3 ([11]): Given $a \in \mathcal{D}$. Mappings $x \mapsto x \backslash\!\!\diamond a$ and $x \mapsto a \diamond\!\!/ x$ are antitone.

III. TEG Model

A Petri net (*e.g.,* [12]) is a tuple (P, T, A, w, x_0) where P is the (finite) set of places, T the (finite) set of transitions, $A \subseteq (P \times T) \cup (T \times P)$ the set of arcs from places to transitions and from transitions to places. $w : (P \times T) \cup (T \times P) \to \mathbb{N}_0$ is a weight function and $(p, t) \in A$ (resp. $(t, p) \in A$) if and only if $w(p, t) \geqslant 1$ (resp. $w(t, p) \geqslant 1$). $x_0 \in \mathbb{N}_0^{|P|}$ is the vector of initial markings (number of tokens). A Petri net can be interpreted as a dynamic system with state space $\mathbb{N}_0^{|P|}$ and initial state x_0. The evolution of the state is governed by the following rules:

1) a transition $t \in T$ can "fire" in state x if $\forall i, x_i \geqslant w(p_i, t)$.
2) if a transition t fires, each element x_i of x changes its value according to $x_i \to x_i - w(p_i, t) + w(t, p_i)$.

Petri nets can be equipped with a notion of time by introducing so-called holding times for the places. They represent

the minimal time a token has to stay in a place before it can contribute to the firing of a transition.

A timed event graph (TEG) is a Petri net with holding times where each place has exactly one input and one output transition and where each arc has weight 1, *i.e.,* $\sum_{i=1}^{|T|} w(p,t_i) = \sum_{i=1}^{|T|} w(t_i,p) = 1$.

For TEGs, the earliest firing times of transitions can be computed as follows:

$$z_i(k) = \max_{j,w(p_j,t_i)=1} (\pi_j(k) + h_j) \tag{5}$$

$$\pi_j(k + x_{0j}) = z_i(k) \text{ with } i \text{ such that } w(t_i,p_j) = 1 \tag{6}$$

where $z_i(k)$ is the earliest time for the k-th firing of transition t_i, $\pi_j(k)$ is the earliest time for place p_j to receive its k-th token, and h_j is the holding time associated with p_j. By eliminating the π_j from the above equation, one gets an (implicit) difference equation for the vector $z(k)$ of the earliest firing times. Clearly, this equation becomes linear in the $(\max,+)$-algebra.

The above relation can be represented even more conveniently in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is formally the quotient dioid of $\mathbb{B}[\![\gamma,\delta]\!]$, the set of formal power series in two variables (γ,δ) with Boolean coefficients and with exponents in $\mathbb{Z}\cup\{-\infty,+\infty\}$, by the equivalence relation $x\mathcal{R}y \Leftrightarrow \gamma^*(\delta^{-1})^*x = \gamma^*(\delta^{-1})^*y$. The dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is complete with the bottom element $\varepsilon = \gamma^{+\infty}\delta^{-\infty}$ and the top element $\top = \gamma^{-\infty}\delta^{+\infty}$.

The resulting relation is now an algebraic equation in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, and the step from a $(\max,+)$-model to a $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$-model is reminiscent of the classical z-transformation. When an element s of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is used to code information concerning a transition of a TEG, then a monomial $\gamma^k\delta^t$ with $k,t \geqslant 0$ may be interpreted as *at most k events ("firings") occur up to time t* or equivalently *event ("firing") $k+1$ occurs at the earliest at time t*. A polynomial is defined as a finite sum of monomials of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

If a TEG behavior is uncertain (number of tokens and time delays are only known to belong to intervals), it can be described in the dioid $I(\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!])$ (the dioid of intervals based on the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$). An element of $I(\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!])$ is a monomial (resp. polynomial) if its lower and upper bounds are monomials (resp. polynomials).

In the following, $\mathcal{D}$ denotes both $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and $I(\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!])$.

In $\mathcal{D}$, a TEG can be described by the following model:

$$\begin{cases} x = Ax \oplus Bu \\ y = Cx \end{cases} \tag{7}$$

where $x \in \mathcal{D}^n$ is the internal state (it represents the earliest firing times of internal transitions), $u \in \mathcal{D}^p$ the input (it represents the earliest firing times of input transitions), and $y \in \mathcal{D}^m$ the output (it represents the earliest firing times of output transitions). Then, $A \in \mathcal{D}^{n\times n}$, $B \in \mathcal{D}^{n\times p}$, and $C \in \mathcal{D}^{m\times n}$.

The transfer function matrix H of the system is calculated with Th. 1:

$$H = CA^*B \tag{8}$$

Each entry of this matrix is a pseudo-periodic element of $\mathcal{D}$. It can be written $H_{ij} = p_{ij} \oplus q_{ij}r_{ij}^*$ where p_{ij} is a polynomial representing the transient part, q_{ij} is a polynomial representing a pattern which is periodically repeated, and $r_{ij} = \gamma^\nu\delta^\tau$ is a monomial corresponding to the periodicity (or throughput), *i.e.,* the pattern q_{ij} will be repeated each ν events and τ time units.

Definition 6 (Strict Periodicity): An element $s \in \mathcal{D}$ is said to be strictly periodic if it can be written as $s = qr^*$ with q a polynomial and r a monomial. q is called pattern of s and r is called throughput of s. A matrix is said to be strictly periodic if all its entries are strictly periodic.

IV. Hierarchical Control

The usual specification for the control of timed event graphs is to achieve a just-in-time policy, *i.e.,* to start all activities as late as possible without making the input/output relation of the controlled system slower than a desired reference model G_{spec}.

To keep exposition reasonably simple, we consider only two control layers (see Fig. 1).

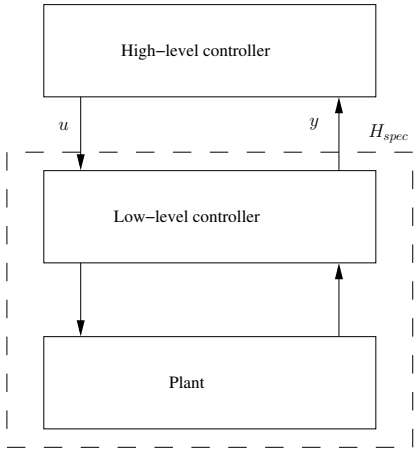

Fig. 1. Hierarchical control with two layers

The synthesis of the low-level controller is not the subject of this paper. We assume it has been designed correctly in the sense that the plant under low-level control satisfies the provided low-level specification H_{spec}. H_{spec} is a model in the dioid $\mathcal{D}$ representing the desired input/output behavior of the plant under low-level control. We say that a specification H_{spec1} is at least as strict as H_{spec2} if $H_{spec1} \preceq H_{spec2}$.

In the following, we investigate conditions for H_{spec} such that there exists a unique optimal high-level controller that achieves the overall specification G_{spec} on the basis of an abstraction H_{spec}. If H_{spec1} and H_{spec2}, with $H_{spec1} \preceq H_{spec2}$, satisfy these conditions, we show that $F_{opt1} \succeq F_{opt2}$, where F_{opt1} (resp. F_{opt2}) is the optimal high-level controller corresponding to H_{spec1} (resp. H_{spec2}).

The cases with or without uncertainties are concurrently discussed ($\mathcal{D}$ denotes both $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and $I(\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!])$). Considerations regarding causality and realizability of the

feedback are omitted. The reader is refered to [6] for details on this topic.

A. Class of Low-level Specification

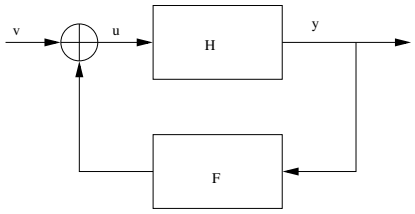

Fig. 2. TEG with an output feedback F

In the following, a linear output feedback controller is considered: $u = Fy \oplus v$ with v an external input. Inserting this control law into the model (7),(8) provides (see Fig. 2):

$$y = HFy \oplus Hv \tag{9}$$

According to Th. 1, the least solution of (9) is $y = (HF)^* Hv$ where $H = CA^*B$ is the open-loop transfer function matrix . Then, the closed-loop transfer function matrix is $(HF)^* H$.

Considering that the transfer function matrix H of the system is known, the following theorem gives a class of specifications G such that the greatest output feedback exists.

Theorem 3: Let $G \in \mathcal{D}^{m\times p}$. The following sets are considered:

- $\mathcal{G}_1 = \{G | \exists D \in \mathcal{D}^{m\times m}$ such that $G = D^*H\}$
- $\mathcal{G}_2 = \{G | \exists D \in \mathcal{D}^{p\times p}$ such that $G = HD^*\}$

If $G_{spec} \in \mathcal{G}_1 \cup \mathcal{G}_2$, there exists a greatest output feedback F_{opt} such that $(HF_{opt})^* H \preceq G_{spec}$ and F_{opt} is given by $H \backslash\!\!\backslash G_{spec} /\!\!/ H$.

This theorem comes from [6] (resp. [13]) considering $\mathcal{D} = \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ (resp. $\mathcal{D} = I(\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!])$).

In the considered approach, the high-level specification G_{spec} is known, but the problem is to find a low-level specification H_{spec} ensuring the existence of the greatest output feedback. It leads to a reformulation of the previous theorem.

Theorem 4: Let $G_{spec} \in \mathcal{D}^{m\times p}, H \in \mathcal{D}^{m\times p}$. The following sets are considered:

- $\mathcal{H}_1 = \{H | \exists D \in \mathcal{D}^{m\times m}$ such that $G_{spec} = D^*H\}$
- $\mathcal{H}_2 = \{H | \exists D \in \mathcal{D}^{p\times p}$ such that $G_{spec} = HD^*\}$

If $H_{spec} \in \mathcal{H}_1 \cup \mathcal{H}_2$, there exists a greatest output feedback F_{opt} such that $(H_{spec}F_{opt})^* H_{spec} \preceq G_{spec}$ and F_{opt} is given by $H_{spec} \backslash\!\!\backslash G_{spec} /\!\!/ H_{spec}$.

In the following, with some assumptions on G_{spec}, a constructive method is presented to find a subset of $\mathcal{H}_1 \cup \mathcal{H}_2$.

Assumption 1: The high-level specification G_{spec} is strictly periodic and all coefficients in the same row (resp. column) have the same throughput. Formally, there exist a matrix $P \in \mathcal{D}^{m\times p}$ with polynomial entries and a diagonal matrix M in $\mathcal{D}^{m\times m}$ (resp. $\mathcal{D}^{p\times p}$) with monomial entries such that $G_{spec} = M^*P$ (resp. $G_{spec} = PM^*$).

Proposition 4: Consider a high-level specification $G_{spec} = M^*P$ fulfilling Ass. 1, then $\{N^*P | N \preceq M\} \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$. Similarly, for $G_{spec} = PM^*$ fulfilling Ass. 1, $\{PN^* | N \preceq M\} \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$.

Proof: As $N \preceq M$, $M^*N^* = M^*$ and $N^*M^* = M^*$ (see Rem. (4)). Consider $H_{spec} = N^*P$ with $N \preceq M$, then $M^*H_{spec} = M^*P = G_{spec}$. Thus, $H_{spec} \in \mathcal{H}_1 \cup \mathcal{H}_2$ and $\{N^*P | N \preceq M\} \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$. The second part of the previous proposition is similarly proved. ■

Thus, a set of low-level specifications ensuring the existence of the greatest output feedback has been found. It represents behavior having the same pattern as the high-level specification but with a different throughput.

B. Trade-Off Between Low-level and High-level Controllers

Considering two low-level specifications H_{spec1}, H_{spec2} with $H_{spec1} \preceq H_{spec2}$. H_{spec1} is then at least as strict as H_{spec2}, as it requires the earliest possible event times to be less than or equal to those of the corresponding ones in H_{spec2}.

Proposition 5: Consider a high-level specification G_{spec} and two low-level specifications H_{spec1} and H_{spec2} in $\mathcal{H}_1 \cup \mathcal{H}_2$ such that $H_{spec1} \preceq H_{spec2}$. The optimal output feedback F_{opt1} associated with H_{spec1} is greater than or equal to F_{opt2}, the optimal output feedback associated with H_{spec2}.

Proof: According to Prop. 3:

$$H_{spec1} \preceq H_{spec2} \Rightarrow H_{spec1} \backslash\!\!\backslash G_{spec} \succeq H_{spec2} \backslash\!\!\backslash G_{spec} \tag{10}$$

$$\Rightarrow F_{opt1} \succeq F_{opt2} \tag{11}$$

■

This result makes the trade-off between low-level and high-level controllers explicit. As H_{spec1} is at least as strict as H_{spec2}, it is intuitive that we have more freedom in the high-level controller associated with H_{spec1} than in the high-level controller associated with H_{spec2}. In consequence, $F_{opt2} \preceq F_{opt1}$, and F_{opt2} will guarantee the overall specification G_{spec} for the low-level abstraction H_{spec1}, *i.e.,* $(H_{spec1}F_{opt2})^* H_{spec1} \preceq G_{spec}$, while F_{opt1} will **not** guarantee G_{spec} for the abstraction H_{spec2}.

V. Example

This example is based on $I(\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!])$. Examples with $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ could be handled in a similar way. The following calculations are done with [14], [15].

The manufacturing system drawn in Fig. 3 is considered.

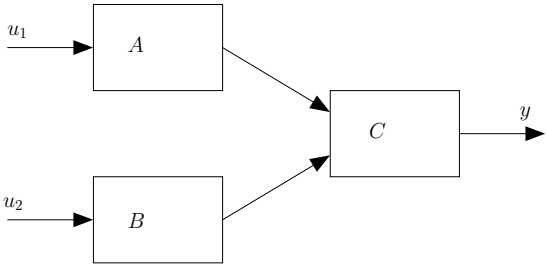

Fig. 3. Schematic Representation of the System

Machine A (resp. B) processes parts with a supply of raw materials in u_1 (resp. u_2). Then, the parts are pairwise assembled in machine C and released in y. The considered inputs for the high-level specification are u_1 and u_2 and the considered output is y. Other inputs are taken care of by the low-level controller. The high-level specification is given below:

$$G_{spec} = \left(\left[\delta^2\left(\gamma\delta^3\right)^*, \delta^2\left(\gamma\delta^6\right)^*\right] \left[\delta^3\left(\gamma\delta^2\right)^*, \delta^3\left(\gamma\delta^4\right)^*\right]\right) \quad (12)$$

For the second element of G_{spec}, the lower bound $\delta^3\left(\gamma\delta^2\right)^*$ and the upper bound $\delta^3\left(\gamma\delta^4\right)^*$ are represented in Fig. 4. The associated interval (*i.e.,* $\left[\delta^3\left(\gamma\delta^2\right)^*, \delta^3\left(\gamma\delta^4\right)^*\right]$) appears in grey. Considering that input u_1 is not limiting (*i.e.,* an infinity of tokens at time $-\infty$) and that input u_2 is a step (*i.e.,* an infinity of tokens at time 0), the first event (numbered 0) occurs at the earliest at time 3, the second event occurs at the earliest at a certain time between 5 and 7 time units, etc.

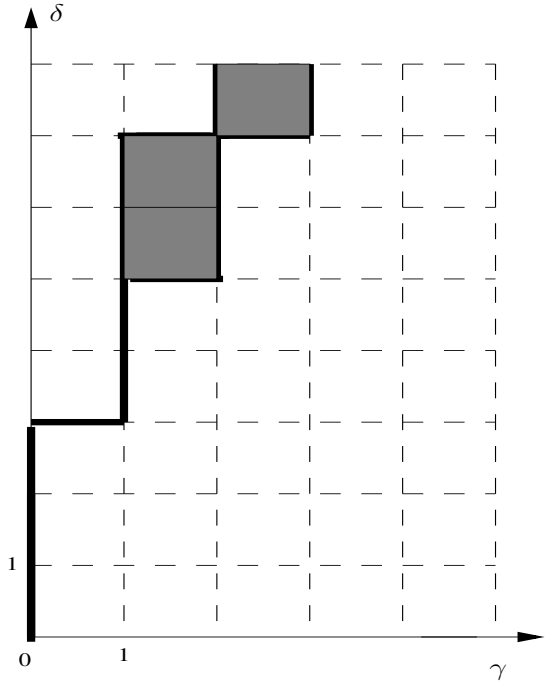

Fig. 4. Upper and lower bound for the second element of G_{spec}

The following low-level specifications are considered:

$$H_{spec1} = \left(\left[\delta^2\left(\gamma\delta\right)^*, \delta^2\left(\gamma\delta^3\right)^*\right] \left[\delta^3\left(\gamma\delta\right)^*, \delta^3\left(\gamma\delta^4\right)^*\right]\right) \quad (13)$$

$$H_{spec2} = \left(\left[\delta^2\left(\gamma\delta^2\right)^*, \delta^2\left(\gamma\delta^5\right)^*\right] \left[\delta^3\left(\gamma\delta^2\right)^*, \delta^3\left(\gamma\delta^4\right)^*\right]\right) \quad (14)$$

These low-level specifications belong to the set of specifications described by Prop. 4. Then, for each low-level specification, a greatest output feedback controller exists. A TEG realization for H_{spec1} (resp. H_{spec2}) is shown in Fig. 5 (resp. Fig. 6).

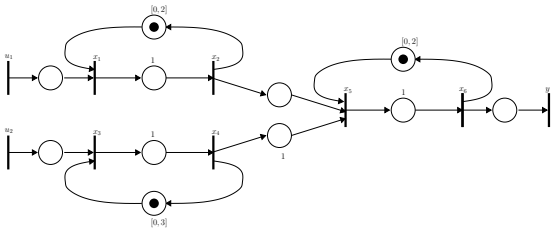

Fig. 5. Realization associated with H_{spec1}

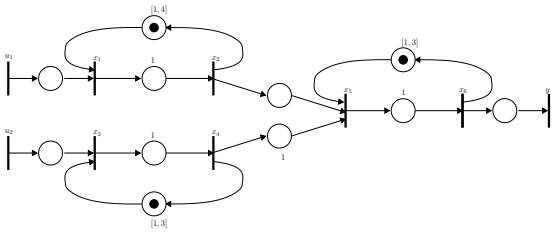

Fig. 6. Realization associated with H_{spec2}

The physical meaning of the previous realizations is obvious: machine A is represented by states x_1, x_2, machine B by states x_3, x_4 and machine C by state x_5, x_6. The following greatest interval output feedbacks are obtained:

$$F_{opt1} = \begin{pmatrix} \left[\gamma\left(\gamma\delta^2\right)^*, \gamma\delta^2\left(\gamma\delta^4\right)^*\right] \\ \left[\gamma^2\delta\left(\gamma\delta^2\right)^*, \gamma\delta\left(\gamma\delta^4\right)^*\right] \end{pmatrix} \quad (15)$$

$$F_{opt2} = \begin{pmatrix} [\varepsilon, \varepsilon] \\ \left[\gamma^2\delta\left(\gamma\delta^2\right)^*, \gamma\delta\left(\gamma\delta^4\right)^*\right] \end{pmatrix} \quad (16)$$

As expected, $F_{opt2} \preceq F_{opt1}$.

VI. Conclusion

In the systematic design of hierarchical control systems, one often encounters a trade-off situation relating the synthesis procedure for adjacent control layers. Typically, this is due to the fact that specifications for low-level control also serve as abstractions for the plant under low-level control and therefore also play an essential role when synthesizing the next (from bottom to top) control layer. In this paper, we have formally investigated this trade-off for the specific case of two control layers, where the top layer's sole task is the timing of certain discrete input events. Consequently, all other decisions (regarding both the ordering of input events and the choice of continuous-valued input signals) are taken by low-level control. Low-level specifications may therefore be realized by timed events graphs, which can be conveniently described using the diod algebra $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ or, if uncertainty is involved,

by using the corresponding interval dioid $I\left(\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]\right)$. Given an overall specification G_{spec}, we first describe a set of feasible low-level specifications H_{spec}. Each element in this set can serve as an abstraction for which a high-level controller F_{opt} exists that enforces G_{spec}. Furthermore, we show that $H_{spec1} \preceq H_{spec2}$ implies that $F_{opt2} \preceq F_{opt1}$, where $\preceq$ is the natural partial order defined in the considered dioid. Hence, a stricter low-level specification H_{spec1} will result in a less restrictive high-level controller F_{opt1}.

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REFERENCES

[1] T. Moor, J. Raisch, and J. M. Davoren, "Admissibility criteria for a hierarchical design of hybrid control systems," in *Proc. ADHS03 IFAC Conference on Analysis and Design of Hybrid Systems*, St. Malo, France, 2003, p. 389394.

[2] J. Raisch and T. Moor, *Hierarchical Hybrid Control of a Multiproduct Batch Plant*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, 2005, vol. 322, p. 99216.

[3] J. Willems, "Models for dynamics." in *Dynamics Reported*, vol. 2, 1989, pp. 172–269.

[4] ——, "Paradigms and puzzles in the theory of dynamic systems," in *IEEE Transactions on Automatic Control*, vol. 36, 1991, pp. 258–294.

[5] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity, An Algebra for Discrete Event Systems*. New York, USA: John Wiley and Sons, 1992.

[6] B. Cottenceau, L. Hardouin, J.-L. Boimond, and J.-L. Ferrier, "Model reference control for timed event graphs in dioids," *Automatica*, vol. 37, no. 9, pp. 1451–1458, Sep. 2001, available at doi:10.1016/S0005-1098(01)00073-5.

[7] G. Cohen, P. Moller, J.-P. Quadrat, and M. Viot, "Algebraic tools for the performance evaluation of discrete event systems," *Proceedings of the IEEE*, vol. 77, no. 1, pp. 39–58, Jan. 1989, special issue on Discrete Event Systems.

[8] G. L. Litvinov and A. N. Sobolevskiĭ, "Idempotent Interval Analysis and Optimization Problems," *Reliable Computing*, vol. 7, no. 5, pp. 353 – 377, 2001, available at arxiv.org/abs/math/0101080.

[9] M. Lhommeau, L. Hardouin, J.-L. Ferrier, and I. Ouerghi, "Interval Analysis in Dioid : Application to Robust Open Loop Control for Timed Event Graphs," in *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, Dec. 2005, pp. 7744–7749, available at doi:10.1109/CDC.2005.1583413.

[10] T. S. Blyth and M. F. Janowitz, *Residuation Theory*. Oxford, United Kingdom: Pergamon press, 1972.

[11] S. Gaubert, "Théorie des Systèmes Linéaires dans les Dioïdes," Thèse, École des Mines de Paris, July 1992.

[12] T. Murata, "Petri Nets: Properties, Analysis and Applications." in *Proceedings of the IEEE*, Apr. 1989, pp. 541–580.

[13] M. Lhommeau, L. Hardouin, B. Cottenceau, and L. Jaulin, "Interval Analysis and Dioid: Application to Robust Controller Design for Timed Event Graphs," *Automatica*, vol. 40, no. 11, pp. 1923–1930, Nov. 2004, available at doi:10.1016/j.automatica.2004.05.013.

[14] B. Cottenceau, L. Hardouin, M. Lhommeau, and J.-L. Boimond, "Data processing tool for calculation in dioid," in *Proceedings of the 5th International Workshop on Discrete Event Systems, WODES 2000*, Ghent, Belgium, 2000, available at istia.univ-angers.fr/ hardouin/outils.html.

[15] L. Hardouin, B. Cottenceau, and M. Lhommeau, "Software tools for manipulating periodic series," 2001, available at istia.univ-angers.fr/ hardouin/outils.html.