Proceedings of the
46th IEEE Conference on Decision and Control
New Orleans, LA, USA, Dec. 12-14, 2007

On the construction of quadratically stable switched linear systems with multiple component systems

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Abstract— **We consider the stability of switched linear systems with multiple component systems of the n−th order. Using ideas from dissipative dynamical systems we construct a set of LTI systems sharing a quadratic Lyapunov function. Switched systems composed of subsystems from this set are exponentially stable for arbitrary switching sequences.**

I. Introduction

The past decade has witnessed an enormous interest in systems whose dynamic behavior can be described mathematically by a combination of continuous dynamics and switching. A typical approach adopted in modelling and analysis is to describe these systems through some switching mechanism that orchestrates among continuous-time dynamical systems. A fundamental question in the analysis of such systems concerns the stability of the resulting switched system under arbitrary switching. It is well known that the equilibrium in a switched linear system with multiple component systems is guaranteed to be exponentially stable under arbitrary switching if and only if there exists a common Lyapunov function for each of the component systems. However, analytic conditions for the existence of such Lyapunov functions are scarce and often concern restricted system classes. The vast majority of such results consider switched systems with only two component systems. In this contribution we present a method to construct a switched system with *more than two component systems* such that all component systems share a quadratic Lyapunov function. We adopt the approach in [8] and cast the stability problem into a behavior theoretical framework to construct a set of behaviors that share the same storage function. We show that the corresponding autonomous dynamics have a common quadratic Lyapunov function.

This paper is organized as follows: We give a concise problem statement in Section II and provide a review of some well known results. The notation used in this paper is summarized in Section III. The treatment in this paper uses tools and ideas from the behavioral theory of dynamical systems, which is introduced briefly in Section IV. Our framework also uses tools from the theory of dissipative systems. We present major ideas from this theory in Section V. Section VI contains the main results of this paper. We present a special case of our result in Section VII and compare it with known results.

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This work was partially supported by the European Union funded HYCON Network of Excellence, contract number FP6-IST-511368.

II. Problem statement and literature review

In this paper we study the stability properties of the switched linear system:

$$\Sigma_S : \dot{x}(t) = A(t)x(t),\ A(t) \in \mathcal{A} = \{A_i, i \in \mathcal{I}\} \quad (1)$$

where $\mathcal{A}$ is a set of $\mathtt{n} \times \mathtt{n}$ matrices indexed by $\mathcal{I}$. The map $t \to A(t)$ is piecewise constant, mapping from $\mathbb{R}^+$ into $\mathcal{A}$.For every such mapping there is a corresponding map σ, from the set of non-negative real numbers to $\mathcal{I}$. The points of discontinuity of $A(t)$ are known as *switching instances*, and the map σ is called a switching signal. We denote the set of all switching signals by $\mathcal{S}$. By a solution of Equation (1) we mean an absolutely continuous function x and a switching signal σ such that $\dot{x}(t) = A_{\sigma(t)}x(t)$ for all t except possibly the switching instances. One can think of the system in Equation (1) as being constructed by switching among linear, time-invariant autonomous systems $\Sigma_{A_i} : \dot{x} = A_i x,\ i \in \mathcal{I}$, which we call the *component systems* of Σ_S. In this paper we consider the following problem:

Given the switched system Σ_S, determine conditions on the matrices $A_i, i \in \mathcal{I}$ such that the equilibrium state 0 in Σ_S is uniformly exponentially stable for every switching signal $\sigma \in \mathcal{S}$.

Note that it is of course *necessary* for the stability of Σ_S that every component system Σ_{A_i} be stable. Stability of every component system is however not *sufficient* to guarantee that the equilibrium in Σ_S is stable under arbitrary switching. Molachanov and Pyatnitskiy [3] show that uniform exponential stability of the equilibrium in Σ_S is equivalent to the existence of a common Lyapunov function $V(x)$ for the component systems Σ_{A_i}, $i \in \mathcal{I}$ and specify a number of properties of this function. However, such non-constructive converse theorems are not directly applicable to check stability for a given system Σ_S. In the last decade many conditions have been derived that guarantee the existence of a common Lyapunov function for a set of LTI systems [2], [1], [16]. The majority of such conditions consider the existence of a common *quadratic* Lyapunov function (CQLF) $V(x) = x^T Lx$ with $L = L^T > 0$ such that the linear matrix inequalities (LMIs) $A_i^T L + LA_i < 0$ are satisfied for all $i \in \mathcal{I}$. Convex optimization tools can be used to check the feasibility of such a set of LMIs. However, this numerical approach fails to give much insight into the stability or instability mechanisms of the system and does not supply any guidelines for designing stable switched systems.

Note that given a set of matrices A_i, $i \in \mathcal{I}$ that satisfy $A_i^T L + LA_i < 0$, $L > 0$, the matrix $\bar{A} = \sum_{i\in\mathcal{I}} \alpha_i A_i$,

$\alpha_i \geq 0$, $\alpha_j \neq 0$ for some $j \in \mathcal{I}$ also satisfies the inequality $\bar{A}^T L + L\bar{A} < 0$. We call the set

$$\mathcal{C} = \{\bar{A} | \exists \alpha_i \geq 0, \text{ not all zero such that } \bar{A} = \sum_{i \in \mathcal{I}} \alpha_i A_i\}$$

the cone generated by A_i, $i \in \mathcal{I}$. The set $\mathcal{C}$ is used in the sequel for characterizing a switched system having an exponentially stable equilibrium under arbitrary switching.

A number of analytic conditions for the existence of a CQLF for several sub-classes of switched systems have been derived in the recent past. If the component systems have some special structure, e.g. if all system matrices A_i are upper triangular (or simultaneously triangularizable) [4], [13], or they commute pairwise [5] then they have a CQLF. Slightly relaxed conditions can be found in [15] where stability is proven for switched systems with matrices that are pair-wise simultaneously transformable to upper triangular form. Though important and interesting, all these results suffer from the shortcoming that the property of simultaneous triangularizability is not robust, and is satisfied for only a small class of systems. Pairwise commutativity poses similar problems. For switched systems with only two component systems more general results are known: if two matrices A_1 and A_2 satisfy $\text{rank}(A_1 - A_2) = 1$, *necessary and sufficient* conditions are given in [14]. Further, if system matrices $A_i \in \mathbb{R}^{2\times 2}$, necessary and sufficient conditions, without requiring $\text{rank}(A_1 - A_2) = 1$ are known [12]. Apart from the rank-difference requirement in the former, and the restriction to dimension two in the latter result, both conditions suffer from the fact that stability of switched systems with only two subsystems (and of course their non-negative combination) can be established.

In this paper we address some of the drawbacks in known results on the existence of CQLFs by formulating the problem in a dissipativity theory framework [19]. Dissipativity theory is a natural generalization of Lyapunov theory. Our treatment is at a fairly high level of generality and, in particular, we do not require any special structure on the constituent matrices A_i. Further, and most importantly, in our framework the matrices A_i are of arbitrary (finite) dimension and switched systems with more than two component systems can be treated.

III. Notation

We denote the field of real numbers by $\mathbb{R}$, and that of complex numbers by $\mathbb{C}$. $\mathbb{R}^{\mathtt{m}}$ denotes the set of column vectors over $\mathbb{R}$ having $\mathtt{m}$ rows. $I_{\mathtt{m}}$ and $0_{\mathtt{m}}$ denote the $\mathtt{m} \times \mathtt{m}$ Identity, and Zero matrices, respectively. $\mathbb{R}^{\mathtt{q}\times\mathtt{m}}$ denotes the set of $\mathtt{q} \times \mathtt{m}$ matrices over $\mathbb{R}$. $\mathbb{R}^{\mathtt{q}\times\mathtt{m}}[D]$ denotes the set of $\mathtt{q} \times \mathtt{m}$ polynomial matrices over $\mathbb{R}$ in the indeterminate D. Given $Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{m}}[D] := \sum_{i=0}^{d} \mathtt{Q}_i D^i$, $\mathtt{Q}_i \in \mathbb{R}^{\mathtt{q}\times\mathtt{m}}$, with $\mathtt{Q}_d$ a nonzero matrix, d is called the degree of Q and is denoted by $\deg Q$. Further, if $\mathtt{Q}_d$ is nonsingular, Q is called a regular polynomial matrix. If $\mathtt{Q}_d = I$, Q is called monic. If all roots of $\det Q = 0$ lie in the open left half complex plane, Q is called Hurwitz. Given two vector spaces $\mathcal{V}_1, \mathcal{V}_2$ and a linear operator $\mathcal{K} : \mathcal{V}_1 \to \mathcal{V}_2$, $\text{Ker}\,\mathcal{K}$ denotes the kernel of $\mathcal{K}$ while $\text{Im}\,\mathcal{K}$ denotes the image of $\mathcal{K}$.

IV. Behavioral theory

In recent years, the *behavioral theory of dynamical systems* has emerged as an alternative to input-output (transfer function or state-space) based system analysis. Tools and ideas from the behavioral approach have been used to address problems in a number of different domains: distributed systems [6], H_∞ control [18], absolute stability [7], supervisory control of hybrid systems [10], to name a few. An introduction to behavioral systems theory can be found in [9].

A behavior is, broadly speaking, a collection of trajectories in a pre-defined function space (e.g. the space of locally integrable functions), characterized by certain *laws*. If these laws are linear and time-invariant, the corresponding behavior is called Linear, Time-invariant (LTI). A linear differential behavior is one in which the behavior can be characterized as the solution set of a family of Ordinary Differential Equations (ODEs). A cornerstone of the behavioral approach is an *image representation*: a LTI system Σ with external variables ("inputs" and "outputs") (u, y) is controllable if and only if it can be represented as the image of a linear differential operator acting on free variables in an appropriate space:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell \qquad (2)$$

where $Q \in \mathbb{R}^{\mathtt{q}\times\mathtt{m}}[D]$, and $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[D]$ are polynomial differential operators. The indeterminate "D" denotes symbolic differentiation. The free variable ℓ, also called a latent variable, is stipulated to lie in $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$, the space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{m}}$. Since ℓ is free, one can assume Q and P to be right coprime without loss of generality. The partition of the system variables as (u, y) in Equation (2) is called an input-output partition, with inputs u and outputs y if Q is square and nonsingular, and the rational function PQ^{-1} is proper.

A system can be given by several representations, in terms of inputs, outputs and internal variables. Internal variables that satisfy an "axiom of state"[11] are called "states", and system representations in terms of these variables are called state representations. A representation is a state representation if and only if it is first-order in terms of states, and zeroth order in terms of inputs and outputs. Given a controllable system, having behavior $\mathcal{B}$ as defined in Equation (2), one can construct a polynomial differential operator $X(\frac{d}{dt})$ such that variables x defined as

$$x = X(\frac{d}{dt}) \begin{bmatrix} u \\ y \end{bmatrix}, \ (u, y) \in \mathcal{B} \qquad (3)$$

are state variables. The operator $X(\frac{d}{dt})$ is called a *state map*. With (u, y) an input-output partition of a behavior $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$, $Q \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[D]$, the span of rows of the polynomial matrix $X(D)$ (over $\mathbb{R}$) is precisely the span of rows r_i (over $\mathbb{R}$) such that $r_i Q^{-1}$ is strictly proper. In particular if $Q(D) = \sum_{i=0}^{d} \mathtt{Q}_i D^i$ is regular, $X(\frac{d}{dt})$ can be defined by the polynomial differential operator

$$X(\frac{d}{dt}) = \begin{bmatrix} I \\ I\frac{d}{dt} \\ \vdots \\ I\frac{d^{d-1}}{dt^{d-1}} \end{bmatrix} \quad (4)$$

It is easy to see that the above state map transforms the system with image representation $\begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$ into a "block-companion" form, and further this state representation is minimal in terms of the number of states among all possible state representations.

V. Dissipative Systems

We review basic properties of dissipative systems in this section. The abstract theory of dissipative systems was introduced by Willems, who in 1972 wrote two seminal papers on the subject [19], [20]. The ideas in these papers have been singularly successful in tieing together concepts from network theory, mechanical systems, thermodynamics, and feedback control theory. The dissipation hypothesis which distinguishes dissipative systems from general dynamical systems results in a fundamental constraint on their dynamical behavior. Consider a system $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$ having behavior $\mathcal{B}$, with $P, Q \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[D]$. Consider a nonsingular matrix $\Phi = \Phi^T \in \mathbb{R}^{2\mathtt{m}\times 2\mathtt{m}}$. We define the action of Φ on $\mathcal{L}_1^{\rm loc}(\mathbb{R}, \mathbb{R}^{2\mathtt{m}})$ by defining the quadratic form Q_Φ:

$$Q_\Phi(u,y) := \begin{bmatrix} u^T & y^T \end{bmatrix} \Phi \begin{bmatrix} u \\ y \end{bmatrix}, \ (u,y) \in \mathcal{L}_1^{\rm loc}(\mathbb{R}, \mathbb{R}^{2\mathtt{m}})$$

Further, $\mathcal{B}$ is called Φ-dissipative if

$$\int_{-\infty}^{\infty} Q_\Phi(u,y)dt \geq 0 \ \ \forall (u,y) \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^{2\mathtt{m}}). \quad (5)$$

In the above inequality $\mathcal{D}(\mathbb{R}, \mathbb{R}^{2\mathtt{m}})$ denotes the space of compactly supported locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{2\mathtt{m}}$. A system $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$ is Φ-dissipative if and only if

$$\begin{bmatrix} Q^T(-i\omega) & P^T(-i\omega) \end{bmatrix} \Phi \begin{bmatrix} Q(i\omega) \\ P(i\omega) \end{bmatrix} \geq 0 \ \ \forall \omega \in \mathbb{R}$$

The function Q_Φ is called a "supply function" and is a measure of the generalized power supplied. Also associated with a dissipative system is a function Q_Ψ, called a "storage function", that satisfies the so called "dissipation inequality":

$$\frac{d}{dt}Q_\Psi(x) \leq Q_\Phi(u,y)$$

where x denotes a set of minimal states for $\mathcal{B}$[17]. Note that there may exist nonzero trajectories along which $\frac{d}{dt}Q_\Psi(x)$ exactly equals the supply Q_Φ. This is undesirable in some problems, especially in stability analysis. Therefore, we define a set of "strictly" dissipative systems:

Definition 5.1: Let $G = PQ^{-1}$ with $P, Q \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[D]$ regular. Given $\Phi = \Phi^T \in \mathbb{R}^{2\mathtt{m}\times 2\mathtt{m}}$, the behavior $\mathcal{B}$ defined as $\text{Im}\begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix}$ is called strictly Φ-dissipative if $\exists \epsilon > 0$ such that $\mathcal{B}$ is $(\Phi - \epsilon I_{2\mathtt{m}})$-dissipative

Note that along every nonzero trajectory in a strictly Φ-dissipative system, there exists Q_Ψ such that $\frac{d}{dt}Q_\Psi(x)$ is strictly less than Q_Φ.

In the sequel, we consider supply functions Q_Φ, $\Phi = \Phi^T$, nonsingular, that have the following structure:

$$\Phi = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix}$$

$\Theta_{1j} \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, $j \in \{1,2\}$. The choice of this structure is motivated by the stability problem considered in this paper. Note in particular that $Q_\Phi(0,y) = 0$ for all $y \in \mathbb{R}^{\mathtt{m}}$. The following theorem investigates under what conditions do there exist positive definite storage functions for strictly Φ-dissipative systems.

Theorem 5.2: Let $\Phi = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix} \in \mathbb{R}^{2\mathtt{m}\times 2\mathtt{m}}$ be nonsingular. Let $\mathcal{B}$ defined as $\text{Im}\begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix}$, PQ^{-1} proper, be strictly Φ-dissipative. If all roots of $\det Q = 0$ lie in the open left half complex plane, every storage function of $\mathcal{B}$ with respect to Q_Φ is a positive definite state function.

Proof: Note that the partion (u,y) is an input-output partition. Consider a minimal state realization (A,B,C,D) of $\mathcal{B}$. Then, $\frac{d}{dt}Q_\Psi(x) < Q_\Phi(u,y)$ along all (x,u,y) that satisfy the realization. Note that A is Hurwitz since Q has all singularities in the open left half complex plane. Then along $u = 0$ we get

$$\frac{d}{dt}Q_\Psi(x) < Q_\Phi(0,y)$$

along all x and y such that $\dot{x} = Ax$, $y = Cx$. Since $Q_\Phi(0,y) = 0 \ \forall y \in \mathbb{R}^{\mathtt{m}}$ it follows that $Q_\Psi(x) > 0$. □

VI. Stability of switched linear systems

We now address the problem of constructing switched linear systems (1) whose component systems Σ_{A_i} have a CQLF. We characterize the component systems in terms of a image representation of a associated strictly dissipative system. This approach, as we shall show, has many advantages. We need several results in order to obtain the characterization mentioned above.

Lemma 6.1: Let $\Phi = \Phi^T = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix}$ be nonsingular with $\Theta_{1j} \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, $j \in \{1,2\}$. Consider a strictly Φ-dissipative behavior $\mathcal{B}$ be defined by $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$, Q regular and Hurwitz, $\deg Q = \deg P$. Let $\mathtt{P}_d$ be the highest degree coefficient of P. Then, $\Theta_{11} + \Theta_{12}^T\mathtt{P}_d + \mathtt{P}_d^T\Theta_{12} > 0$.

Proof Q can be assumed to be monic without loss of generality. Since $\mathcal{B}$ is strictly Φ-dissipative:

$$\Pi(\omega) := Q^T(-i\omega)\Theta_{11}Q(i\omega) + Q^T(-i\omega)\Theta_{12}^T P(i\omega) + P^T(-i\omega)\Theta_{12}Q(i\omega) \geq \epsilon(Q^T(-i\omega)Q(i\omega) + P^T(-i\omega)P(i\omega))$$

for all $\omega \in \mathbb{R}$ and some $\epsilon > 0$. $\Pi(\omega)$ has degree $2\deg Q$. Dividing left, and right hand sides of the inequality by

$\omega^{2\deg Q}$ and taking the limit as $\omega \to \infty$ shows that $\Theta_{11} + \Theta_{12}^T \mathtt{P}_d + \mathtt{P}_d^T \Theta_{12} > 0$. □

Given a supply function Q_Φ, the following Lemma provides for the construction of another associated supply function that will be used in the sequel:

Lemma 6.2: Let $\Phi_1 = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix}$ and $S = \begin{bmatrix} I_{\mathtt{m}} & K \\ 0_{\mathtt{m}} & I_{\mathtt{m}} \end{bmatrix} \in \mathbb{R}^{2\mathtt{m}\times 2\mathtt{m}}$ with $\begin{bmatrix} K^T & I \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} \geq 0$. Define $\Phi_2 = S^{-T}\Phi_1 S^{-1}$. Then $Q_{\Phi_2}(0,y) \leq 0$ for all $y \in \mathbb{R}^{\mathtt{m}}$.

Proof We explicitly write out Φ_2 in terms of Θ_{1j}, $j \in \{1,2\}$ and K :

$$\Phi_2 = \begin{bmatrix} \Theta_{11} & -\Theta_{11}K + \Theta_{12}^T \\ -K^T\Theta_{11} + \Theta_{12} & K^T\Theta_{11}K - K^T\Theta_{12}^T - \Theta_{12}K \end{bmatrix}$$

Since $\begin{bmatrix} K^T & I \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} \geq 0$ it follows that $Q_{\Phi_2}(0,y) \leq 0 \ \forall y \in \mathbb{R}^{\mathtt{m}}$ □

Consider a supply function Q_{Φ_1} and a Φ_1-dissipative behavior $\mathcal{B}_1$. Using the construction of the supply function Q_{Φ_2} in Lemma 6.2, we construct a Φ_2-dissipative behavior having the same storage functions as $\mathcal{B}_1$ with respect to Q_{Φ_1}:

Theorem 6.3: Let Φ_1 and Φ_2 be such that they satisfy conditions in Lemma 6.2. Let $\mathcal{B}_1 = \begin{bmatrix} Q_1(\frac{d}{dt}) \\ P_1(\frac{d}{dt}) \end{bmatrix} \ell$, with Q_1 regular and Hurwitz, $P_1 Q_1^{-1}$ proper, be strictly Φ_1-dissipative . Define $\mathcal{B}_2 = \begin{bmatrix} Q_2(\frac{d}{dt}) \\ P_2(\frac{d}{dt}) \end{bmatrix} \ell$ where

$$\begin{bmatrix} Q_2(D) \\ P_2(D) \end{bmatrix} = S \cdot \begin{bmatrix} Q_1(D) \\ P_1(D) \end{bmatrix}$$

Then, $\mathcal{B}_2$ has the following properties:

1) Q_2 is regular and $\deg Q_2 = \deg Q_1$.
2) Q_2, P_2 are right coprime.
3) $\mathcal{B}_2$ is strictly Φ_2-dissipative.
4) Every storage function (on states) of $\mathcal{B}_2$ with respect to Q_{Φ_2} is also a storage function (on states) of $\mathcal{B}_1$ with respect to Q_{Φ_1}.

Proof

1) Without loss of generality Q_1 can be assumed monic. The case when $P_1 Q_1^{-1}$ is strictly proper is obvious. Let $\deg Q_1 = \deg P_1$. Let $\mathtt{Q}_{2d}$ and $\mathtt{P}_{1d}$ be the highest degree coefficients of Q_2 and P_1 respectively. Then, $\mathtt{Q}_{2d} = I + K\mathtt{P}_{1d}$ since Q_1 is monic by assumption. Suppose Q_2 is not regular. Then there exists nonzero $v \in \mathbb{R}^{\mathtt{m}}$ such that
$$(I_{\mathtt{m}} + K\mathtt{P}_{1d})v = 0$$
or $v = -K\mathtt{P}_{1d}v$. Since by assumption
$$\begin{bmatrix} K^T & I \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} \geq 0$$
we have:
$$v^T \mathtt{P}_{1d}^T(-K^T\Theta_{11}K + K^T\Theta_{12}^T + \Theta_{12}K)\mathtt{P}_{1d}v \geq 0$$
and therefore $-v^T(\Theta_{11} + \Theta_{12}^T\mathtt{P}_{1d} + \mathtt{P}_{1d}^T\Theta_{12})v \geq 0$. However, from Lemma 6.1, $(\Theta_{11} + \Theta_{12}^T\mathtt{P}_{1d} + \mathtt{P}_{1d}^T\Theta_{12}) > 0$, and therefore $v = 0$, which is a contradiction. Hence, $\mathtt{Q}_{2d} = I + K\mathtt{P}_{1d}$ is nonsingular.
2) We show that if Q_2 and P_2 are not right coprime, S is singular, which is a contradiction. If Q_2 and P_2 are not right coprime, there exists $\lambda \in \mathbb{C}$ and nonzero $v \in \mathbb{C}^{\mathtt{m}}$ such that $\begin{bmatrix} Q_2(\lambda) \\ P_2(\lambda) \end{bmatrix} v = 0$. Since Q_1, P_1 can be taken to be right coprime without loss of generality, it follows that $\begin{bmatrix} Q_1^T(\lambda) & P_1^T(\lambda) \end{bmatrix}^T$ has full column rank. Further,
$$\begin{bmatrix} Q_2(\lambda) \\ P_2(\lambda) \end{bmatrix} v = S \begin{bmatrix} Q_1(\lambda) \\ P_1(\lambda) \end{bmatrix} v = 0$$
which shows that S is singular. This is a contradiction to the assumptions in the theorem. Hence, Q_2 and P_2 are right coprime.
3) Note that $S^T\Phi_2 S = \Phi_1$. By definition, $\mathcal{B}_2 = S \cdot (\mathcal{B}_1)$. Since $\mathcal{B}_1$ is strictly Φ_1-dissipative, $\mathcal{B}_2$ is strictly Φ_2-dissipative.
4) Using the concept of a Quadratic Differential Form (QDF) [21] we can show that Q_{Φ_1} along $\mathcal{B}_1$ and Q_{Φ_2} along $\mathcal{B}_2$ are equal on a suitably chosen latent variable ℓ. Define the following polynomial matrices in two variables:
$$\Phi_2' = \begin{bmatrix} Q_2^T(\zeta) & P_2^T(\zeta) \end{bmatrix} \Phi_2 \begin{bmatrix} Q_2(\eta) \\ P_2(\eta) \end{bmatrix} \quad (6)$$
$$\text{and } \Phi_1' = \begin{bmatrix} Q_1^T(\zeta) & P_1^T(\zeta) \end{bmatrix} \Phi_1 \begin{bmatrix} Q_1(\eta) \\ P_1(\eta) \end{bmatrix}$$
Note that Φ_1' can be written as $\sum_{i,j=1}^k \Phi_{1ij}'\zeta^i\eta^j$. We associate a monomial $\Phi_{1ij}'\zeta^i\eta^j$ of Φ_1' with the differential operator $(\frac{d^i}{dt^i})^T\Phi_{1ij}'(\frac{d^j}{dt^j})$ and define $Q_{\Phi_1'}(\ell) = \sum_{ij=0}^k (\frac{d^i\ell}{dt^i})^T\Phi_{1ij}'(\frac{d^j\ell}{dt^j})$. We can define $Q_{\Phi_2'}$ analogously.
Since $\Phi_1' = \Phi_2'$ it follows that $Q_{\Phi_1'}(\ell) = Q_{\Phi_2'}(\ell)$ $\forall \ell \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$. Since $\mathcal{B}_1$ is strictly Φ_1-dissipative it follows that there exists a storage function Q_Ψ:
$$\frac{d}{dt}Q_\Psi(\ell) < Q_{\Phi_1'}(\ell), \quad (7)$$
$$\text{and therefore also } \frac{d}{dt}Q_\Psi(\ell) < Q_{\Phi_2'}(\ell)$$
Since $\deg Q_2 = \deg Q_1$ and Q_2 is regular, it follows that minimal state representations of both $\mathcal{B}_2$ and $\mathcal{B}_1$ have the same dimension. Let $\deg Q_2 = d$. Define
$$X(\frac{d}{dt}) = \begin{bmatrix} I_{\mathtt{m}} \\ I_{\mathtt{m}}\frac{d}{dt} \\ \vdots \\ I_{\mathtt{m}}\frac{d^{d-1}}{dt^{d-1}} \end{bmatrix}$$
Then, $x = X(\frac{d}{dt})\ell$ defines a minimal set of states for both $\mathcal{B}_2$ and $\mathcal{B}_1$. Since Q_Ψ is a state function of $\mathcal{B}_1$ it can be written as $Q_\Psi(\ell) = (X(\frac{d}{dt})\ell)^T LX(\frac{d}{dt})\ell$ where $L = L^T \in \mathbb{R}^{d\times d}$. This shows that L defines a storage

function on states for $\mathcal{B}_1$ with respect to Q_{Φ_1} and also for $\mathcal{B}_2$ with respect to Q_{Φ_2}. □

Having established the existence of a common storage function on states for $\mathcal{B}_1$ and $\mathcal{B}_2$ we now present the following stability theorem for state representations of $\mathcal{B}_1$ and $\mathcal{B}_2$ obtained through the same state map:

Theorem 6.4: Let $\Phi_1 = \Phi_1^T = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix} \in \mathbb{R}^{2\mathtt{m}\times 2\mathtt{m}}$ be nonsingular. Consider a strictly Φ_1-dissipative behavior $\mathcal{B}_1 = \begin{bmatrix} Q_1(\frac{d}{dt}) \\ P_1(\frac{d}{dt}) \end{bmatrix} \ell$, Q_1 regular and Hurwitz. Define $Q_2(D) = \begin{bmatrix} I_{\mathtt{m}} & K \end{bmatrix} \begin{bmatrix} Q_1(D) \\ P_1(D) \end{bmatrix}$ where K satisfies

$$\begin{bmatrix} K^T & I \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} \geq 0. \quad (8)$$

Let $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ be state representations for Ker$Q_1(\frac{d}{dt})$ and Ker$Q_2(\frac{d}{dt})$, respectively, obtained from the same state map. Then, the equilibrium state in $\dot{x} = A(t)x(t)$, $A(t) \in \mathcal{A} = \{A_1, A_2\}$ is uniformly exponentially stable under arbitrary switching.

Proof Define $S = \begin{bmatrix} I_{\mathtt{m}} & K \\ 0_{\mathtt{m}} & I_{\mathtt{m}} \end{bmatrix}$. Then, S is nonsingular. Define $\Phi_2 = S^{-T}\Phi_1 S^{-1}$. Let $P_2(D) = P_1(D)$. Then, $\mathcal{B}_2$ defined as Im $\begin{bmatrix} Q_2(\frac{d}{dt}) \\ P_2(\frac{d}{dt}) \end{bmatrix}$ is strictly Φ_2-dissipative and has the following properties (Theorem 6.3):

1) Q_2 is regular, and $\deg Q_2 = \deg Q_1$.
2) Q_2 and P_2 are right coprime.
3) Every storage function (on states) of $\mathcal{B}_2$ with respect to Q_{Φ_2} is also a storage function (on states) of $\mathcal{B}_1$ with respect to Q_{Φ_1}.

Since Q_1 is Hurwitz and $\mathcal{B}_1$ is strictly Φ_1 dissipative, it follows from Theorem 5.2 that there exists $Q_\Psi(x) > 0$ that satisfies:

$$\frac{d}{dt} Q_\Psi(x) < Q_{\Phi_1}(u_1, y_1) \quad (9)$$

along all (x, u_1, y_1) corresponding to a minimal state representation of $\mathcal{B}_1$, say (A_1, B_1, C_1, F_1). Further since Q_{Φ_1} along $\mathcal{B}_1$ and Q_{Φ_2} along $\mathcal{B}_2$ are equal (on suitably constructed latent variables) it follows from Theorem 6.3 that Q_Ψ also satisfies

$$\frac{d}{dt} Q_\Psi(x) < Q_{\Phi_2}(u_2, y_2) \quad (10)$$

along all (x, u_2, y_2) corresponding to a minimal state representation of $\mathcal{B}_2$, say (A_2, B_2, C_2, F_2), obtained using the same state map as (A_1, B_1, C_1, F_1) in inequality (9). We now consider the autonomous part of state representations for $\mathcal{B}_2$ and $\mathcal{B}_1$ by setting $u_2 = 0$ and $u_1 = 0$. We get:

$$\frac{d}{dt} Q_\Psi(x) < Q_{\Phi_1}(0, y_1)$$

along $\dot{x} = A_1 x, y = C_1 x$ and

$$\frac{d}{dt} Q_\Psi(x) < Q_{\Phi_2}(0, y_2)$$

along $\dot{x} = A_2 x, y_2 = C_2 x$. By construction $Q_{\Phi_1}(0, y_1) = 0$ and $Q_{\Phi_2}(0, y_2)$ is negative-semidefinite (Lemma 6.2). Hence $Q_\Psi(x) = x^T L x$ is a CQLF for Σ_{A_1} and Σ_{A_2} and $A_i^T L + L A_i < 0$, $i \in \{1, 2\}$. □

Note that Theorem 6.4 provides the construction of a switched system (1) with *two modes of arbitrary (finite) order* whose system matrices A_i have a CQLF. We shall now extend the result to a family of autonomous LTI systems parametrized by K.

Note that in particular $\Phi_1 = \Phi_2$, with $K = 0_{\mathtt{m}}$ is admissible in Theorem 6.4. Given $\Phi = \Phi^T = \begin{bmatrix} \Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix}$, nonsingular, and a strictly Φ-dissipative $\mathcal{B}$ corresponding to the proper rational function PQ^{-1}, Q regular and Hurwitz, we define the set $\mathcal{Q}_{\mathcal{B}}$ as follows:

$$\mathcal{Q}_{\mathcal{B}} = \{Q_i \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[D] \mid \exists K,\ \begin{bmatrix} K^T & I_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} -\Theta_{11} & \Theta_{12}^T \\ \Theta_{12} & 0_{\mathtt{m}} \end{bmatrix} \begin{bmatrix} K \\ I_{\mathtt{m}} \end{bmatrix} \geq 0 \text{ such that } Q_i = \begin{bmatrix} I_{\mathtt{m}} & K \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix}\}.$$

Note that $\mathcal{Q}_{\mathcal{B}}$ is non-empty and has at least the element Q. Further, $\forall\, Q_i \in \mathcal{Q}_{\mathcal{B}}$, $\deg Q_i = \deg Q$, and Q_i is regular. We denote by the set $\mathcal{A}$ minimal state representations for Ker $Q_i(\frac{d}{dt})$, $Q_i \in \mathcal{Q}_{\mathcal{B}}$, obtained through the same state map:

$$\mathcal{A} = \{A_i \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \,|\, \dot{x} = A_i x \text{ is a state representation for } \operatorname{Ker} Q_i(\frac{d}{dt}), Q_i \in \mathcal{Q}_{\mathcal{B}}, \text{using the same state map}\} \quad (11)$$

Then, $\mathcal{A}$ is a family of matrices that satisfy a CQLF. The above discussion thus leads to the following result:

Theorem 6.5: Consider the set $\mathcal{A}$ defined in Equation (11). The origin in the switched system Σ_S with component systems $\Sigma_{A_i} : \dot{x} = A_i x$, $A_i \in \mathcal{A}$ is uniformly exponentially stable under arbitrary switching

VII. Special case: Passive Systems

We now examine Theorem 6.4 in greater detail and derive interesting special cases. Define $\Phi_1 = \begin{bmatrix} 0 & I_{\mathtt{m}} \\ I_{\mathtt{m}} & 0 \end{bmatrix}$. Consider a behavior $\mathcal{B}$ defined as $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell$ with Q regular and Hurwitz. If $\mathcal{B}$ is strictly Φ_1-dissipative then the corresponding system is called *passive*, and the rational function PQ^{-1} is strictly positive real. With $Q_{\Phi_1} = \begin{bmatrix} 0 & I_{\mathtt{m}} \\ I_{\mathtt{m}} & 0 \end{bmatrix}$, Theorem 6.4, along with Theorem 6.5 leads to the following corollary:

Corollary 7.1: With PQ^{-1} Strictly Positive Real, define

$$\mathcal{Q}_{\mathcal{B}} = \{Q_i \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[D] \mid \exists K, K + K^T \geq 0 \text{ such that } Q_i = \begin{bmatrix} I_{\mathtt{m}} & K \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix}\}$$

$$\mathcal{A} = \{A_i \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \,|\, \dot{x} = A_i x \text{ is a state representation for } \operatorname{Ker} Q_i(\frac{d}{dt}), Q_i \in \mathcal{Q}_{\mathcal{B}}, \text{using the same state map}\}.$$

The equilibrium state in the switched system Σ_S with component systems $\Sigma_{A_i} : \dot{x} = A_i x$, $A_i \in \mathcal{A}$ is uniformly exponentially stable under arbitrary switching.

We illustrate the result in Corollary 7.1 by an example:

Example 7.2: We define

$$Q_1 = \begin{pmatrix} 1+D & 10 & D \\ 0 & 7+D & 1 \\ 0 & 0 & 8+D \end{pmatrix}$$
$$P_1 = \begin{pmatrix} 2+D & 0 & 0 \\ 0 & 9+D & 3 \\ 1 & 0 & 11+D \end{pmatrix}.$$

Then, $P_1Q_1^{-1}$ is Strictly Positive Real. We define two matrices Q_2, Q_3 using different choices of K in Corollary 7.1: Let $K_2 = I_3$, then

$$Q_2 = Q_1 + K_2P_1 = \begin{pmatrix} 3+2D & 10 & D \\ 0 & 16+2D & 4 \\ 1 & 0 & 19+2D \end{pmatrix},$$

and $K_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ yields

$$Q_3 = Q_1 + K_3P_1 = \begin{pmatrix} 6+2D & 28+2D & 39+4D \\ 10+2D & 43+5D & 79+6D \\ 15+3D & 54+6D & 125+10D \end{pmatrix}.$$

The equations $Q_i(\frac{d}{dt})x = 0$, $i = 1,2,3$ represent autonomous differential-algebraic systems. The highest-degree coefficients of Q_2 and Q_3 are guaranteed to be nonsingular, as shown in Theorem 6.3. Transforming $\text{Ker}Q_i(\frac{d}{dt})$ in the standard state-space form, we get matrices A_i, $i = 1,2,3$:

$$A_1 = \begin{pmatrix} -1 & -10 & 8 \\ 0 & -7 & -1 \\ 0 & 0 & -8 \end{pmatrix},\quad A_2 = \begin{pmatrix} -1.2 & -5 & 4.8 \\ 0 & -8 & -2 \\ -0.5 & 0 & -9.5 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -0.3 & -11 & 12 \\ -0.7 & -6 & -4.5 \\ -1 & 1.5 & -13 \end{pmatrix}.$$

According to Corollary 7.1 the LTI systems associated with $\{A_1, A_2, A_3\}$ have a CQLF. Indeed, it can be verified that:

$$L = \begin{pmatrix} 1.6 & -1.8 & 1.9 \\ -1.8 & 3.8 & -2.6 \\ 1.9 & -2.6 & 3.2 \end{pmatrix}$$

satisfies $L = L^T > 0$ and $A_i^TL + LA_i < 0$, for $i = 1,2,3$.

Note that $\text{rank}(A_i - A_j) = 3$, $\forall\ i,j \in \{1,2,3\}$, $i \neq j$. This is a generalization over results obtained, for instance in [14] where rank of the difference is stipulated to be exactly 1. Note that A_i,$i \in \{1,2,3\}$ are linearly independent (in $\mathbb{R}^9$). Therefore A_3 *does not* lie in the cone generated by A_1, A_2. Thus we have successfully constructed a cone of matrices having a CQLF, with three generators. We can further enlarge this cone by adding more generators by an appropriate choice of the parameter matrix K.

VIII. Conclusion

In this paper we have presented a recipe to construct a set of linear autonomous systems that have a common quadratic Lyapunov function. A switched system having component systems from this set has a uniformly exponentially stable equilibrium under arbitrary switching. This set is constructed from an associated dissipative dynamical system. The approach presented here extends previous stability results on switched systems in several directions: firstly, the component systems may be of arbitrary finite order without further structural restrictions (e.g. the restriction of being triangular, or in companion form); secondly, we present a method to construct a cone of matrices with more than two generators such that all matrices inside the cone satisfy a CQLF.

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