Chapter 22
Discrete-Event Systems in a Dioid Framework: Control Theory

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22.1 Motivation

This chapter deals with the control of discrete-event systems which admit linear models in dioids (or idempotent semirings), introduced in the previous chapter (see also the books [1, 19]). These systems are characterized by synchronization and delay phenomena. Their modeling is focused on the evaluation of the occurrence dates of events. The outputs correspond to the occurrence dates of events produced by the system and the inputs are the occurrence dates of exogenous events acting on it. The control of the system inputs will lead to a scheduling of these occurrence dates such that a specified objective for the system be achieved. For example, in a manufacturing management context, the controller will have to schedule the input dates of raw materials or the starting of jobs in order to obtain required output dates for the produced parts.

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The nature of the considered systems is such that they cannot be accelerated, indeed the fastest behavior of the system is fixed and known. The only degree of freedom is to delay the inputs. For example, in a manufacturing system, the maximal throughput of a machine is obtained with its fastest behavior, hence the only thing which can be done is to delay the starting of jobs or decrease its production speed. Hence, the control synthesis presented in this part will aim at controlling the input dates in the system in order to get the outputs at the specified date. More precisely, it will be shown that the proposed control strategies will aim at delaying the inputs as much as possible while ensuring that the outputs occur before the specified dates. This kind of strategy is rather popular in an industrial context, since it is optimal with regard to the just-in-time criterion, which means that the jobs will start as late as possible, while ensuring they will be completed before the required date. The optimization of this criterion means that the parts will spend a minimal time in the system, *i.e.*, it will avoid useless waiting time. This strategy is worthwhile from an economic point of view.

This chapter will be organized as follows. In a first step, some theoretical results will be given; they come in addition to the ones introduced in Chapter 21. Then the control strategies will be presented, starting with open-loop strategies and concluding with closed-loop ones. For both, an illustration based on the example introduced in Chapter 21 will be provided.

22.2 Theory and Concepts

The systems considered and the algebraic context were described in the previous chapter. Classically, the control of a system has to do with system inversion. Hence we recall in this section several algebraic results allowing us to invert linear equations in dioids, *i.e.*, we focus on the resolution of a system $\boldsymbol{A} \otimes \boldsymbol{X} = \boldsymbol{B}$, where $\boldsymbol{A}$, $\boldsymbol{X}$ and $\boldsymbol{B}$ are matrices with entries in a given dioid. The product law of semirings does not necessarily admit an inverse, nevertheless it is possible to consider residuation theory to get a pseudo inverse. This theory (see [3]) allows us to deal with inversion of mappings defined over ordered sets. In our algebraic framework, it will be useful to get the greatest solution of the inequality $\boldsymbol{A} \otimes \boldsymbol{X} \preceq \boldsymbol{B}$. Indeed, a dioid is an ordered set, because the idempotency of the addition law defines a canonical order ($a \oplus b = a \Leftrightarrow a \succeq b$). Hence this theory is suitable to invert mappings defined over dioids. The main useful results are recalled in the first part of the chapter. Another useful key point is the characterization of solutions of implicit equations such as $x = a \otimes x \oplus b$. It will be recalled in a second step that the latter admits a smallest solution, namely a^*b, where $*$ is the Kleene star operator. Some useful properties involving this operator will then be recalled.

22.2.1 Mapping Inversion Over a Dioid

Definition 22.1 (Isotone mapping). *An order-preserving mapping $f : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ and $\mathcal{C}$ are ordered sets, is a mapping such that: $x \succeq y \Rightarrow f(x) \succeq f(y)$. It is said to be* ***isotone*** *in the sequel.*

Definition 22.2 (Residuated mapping). *Let $\mathcal{D}$ and $\mathcal{C}$ be two ordered sets. An isotone mapping $\Pi : \mathcal{D} \to \mathcal{C}$ is said to be residuated, if the inequality $\Pi(x) \preceq b$ has a greatest solution in $\mathcal{D}$ for all $b \in \mathcal{C}$.*

The following theorems yield necessary and sufficient conditions to characterize these mappings.

Theorem 22.1. [2, 1] *Let $\mathcal{D}$ and $\mathcal{C}$ be two ordered sets. Let $\Pi : \mathcal{D} \to \mathcal{C}$ be an isotone mapping. The following statements are equivalent.*

(i) Π is residuated.
(ii) There exists an isotone mapping denoted $\Pi^\sharp : \mathcal{C} \to \mathcal{D}$ such that $\Pi \circ \Pi^\sharp \preceq \mathsf{Id}_\mathcal{C}$ and $\Pi^\sharp \circ \Pi \succeq \mathsf{Id}_\mathcal{D}$.

Theorem 22.2. *Let $\mathcal{D}$ and $\mathcal{C}$ be two ordered sets. Let $\Pi : \mathcal{D} \to \mathcal{C}$ be a residuated mapping; then the following equalities hold:*

$$\begin{aligned} \Pi \circ \Pi^\sharp \circ \Pi &= \Pi \\ \Pi^\sharp \circ \Pi \circ \Pi^\sharp &= \Pi^\sharp \end{aligned} \tag{22.1}$$

Example 22.1. In [1], the following mappings are considered:

$$\begin{aligned} L_a &: \mathcal{D} \to \mathcal{D} \\ &\quad x \mapsto a \otimes x \quad \text{(left product by } a\text{)}, \\ R_a &: \mathcal{D} \to \mathcal{D} \\ &\quad x \mapsto x \otimes a \quad \text{(right product by } a\text{)}, \end{aligned} \tag{22.2}$$

where $\mathcal{D}$ is a dioid. It can be shown that these mappings are residuated. The corresponding residual mappings are denoted:

$$\begin{aligned} L_a^\sharp(x) &= a \backslash\!\!\circ\, x \quad \text{(left division by } a\text{)}, \\ R_a^\sharp(x) &= x \,/\!\!\circ\, a \quad \text{(right division by } a\text{)}. \end{aligned} \tag{22.3}$$

■

Therefore, equation $a \otimes x \preceq b$ has a greatest solution denoted $L_a^\sharp(b) = a \backslash\!\!\circ\, b$. In the same way, $x \otimes a \preceq b$ admits $R_a^\sharp(b) = b \,/\!\!\circ\, a$ as its greatest solution. Let $\boldsymbol{A}, \boldsymbol{D} \in \mathcal{D}^{m \times n}$, $\boldsymbol{B} \in \mathcal{D}^{m \times p}$, $\boldsymbol{C} \in \mathcal{D}^{n \times p}$ be some matrices. The greatest solution of inequality $\boldsymbol{A} \otimes \boldsymbol{X} \preceq \boldsymbol{B}$ is given by $\boldsymbol{C} = \boldsymbol{A} \backslash\!\!\circ\, \boldsymbol{B}$ and the greatest solution of $\boldsymbol{X} \otimes \boldsymbol{C} \preceq \boldsymbol{B}$ is given by $\boldsymbol{D} = \boldsymbol{B} \,/\!\!\circ\, \boldsymbol{C}$. The entries of matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are obtained as follows:

$$\begin{aligned} \boldsymbol{C}_{ij} &= \bigwedge_{k=1}^{m} (\boldsymbol{A}_{ki} \backslash\!\!\!\circ \boldsymbol{B}_{kj}) \\ \boldsymbol{D}_{ij} &= \bigwedge_{k=1}^{p} (\boldsymbol{B}_{ik} \,/\!\!\!\circ\, \boldsymbol{C}_{jk}), \end{aligned} \tag{22.4}$$

where $\wedge$ is the greatest lower bound. It must be noted that $\forall a \in \mathscr{D}$, $\varepsilon \otimes x \preceq a$ admits $\varepsilon \backslash\!\!\!\circ a = \top$ as its greatest solution. Indeed, ε is absorbing for the product law. Furthermore, $\top \otimes x \preceq a$ admits $\top \backslash\!\!\!\circ a = \varepsilon$ as a solution, except if $a = \top$. Indeed, in the latter case $\top \backslash\!\!\!\circ \top = \top$, that is to say $\top \otimes x \preceq \top$ admits $\top$ as a greatest solution.

Example 22.2. Let us consider the following relation $\boldsymbol{AX} \preceq \boldsymbol{B}$ with $\boldsymbol{A}^t = \begin{bmatrix} 2 & \varepsilon & 0 \\ 5 & 3 & 7 \end{bmatrix}$ and $\boldsymbol{B}^t = \begin{bmatrix} 6 & 4 & 9 \end{bmatrix}$ being matrices with entries in $\overline{\mathbb{Z}}_{\max}$. By recalling that in this dioid the product corresponds to the classical sum, the residuation corresponds to the classical minus. That is to say, $5 \otimes x \preceq 8$ admits the solutions set $\mathscr{X} = \{x | x \preceq 5 \backslash\!\!\!\circ 8, \text{ with } 5 \backslash\!\!\!\circ 8 = 8 - 5 = 3\}$, with 3 being the greatest solution of this set[1]. Hence, by applying the computation rules of Equation (22.4), the following result is obtained:

$$\boldsymbol{C} = \boldsymbol{A} \backslash\!\!\!\circ \boldsymbol{B} = \begin{bmatrix} 2 \backslash\!\!\!\circ 6 \wedge \varepsilon \backslash\!\!\!\circ 4 \wedge 0 \backslash\!\!\!\circ 9 \\ 5 \backslash\!\!\!\circ 6 \wedge 3 \backslash\!\!\!\circ 4 \wedge 7 \backslash\!\!\!\circ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

This matrix $\boldsymbol{C}$ is the greatest such that $\boldsymbol{A} \otimes \boldsymbol{C} \preceq \boldsymbol{B}$. See the verification below.

$$\boldsymbol{A} \otimes (\boldsymbol{A} \backslash\!\!\!\circ \boldsymbol{B}) = \begin{bmatrix} 2 & 5 \\ \varepsilon & 3 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix} \preceq \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix} = \boldsymbol{B}.$$

According to Theorem 22.2, the reader is invited to check that:

$$\boldsymbol{A} \backslash\!\!\!\circ \big(\boldsymbol{A} \otimes (\boldsymbol{A} \backslash\!\!\!\circ \boldsymbol{B})\big) = \boldsymbol{A} \backslash\!\!\!\circ \boldsymbol{B}. \tag{22.5}$$

Furthermore, it can be noticed that equation $\boldsymbol{A} \otimes \boldsymbol{X} = \boldsymbol{B}$ admits a greatest solution if matrix $\boldsymbol{B}$ is in the image of matrix $\boldsymbol{A}$, which is equivalent to say that there exists a matrix $\boldsymbol{L}$ such that $\boldsymbol{B} = \boldsymbol{A} \otimes \boldsymbol{L}$. By considering the matrix

$$\boldsymbol{B} = \boldsymbol{A} \otimes \boldsymbol{L} = \begin{bmatrix} 2 & 5 \\ \varepsilon & 3 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 8 \end{bmatrix},$$

the reader is invited to check that $\boldsymbol{A} \otimes (\boldsymbol{A} \backslash\!\!\!\circ \boldsymbol{B}) = \boldsymbol{B}$. ■

As in the classical algebraic context, in the absence of ambiguity, the operator $\otimes$ will be omitted in the sequel.

Below, some properties about this "division" operator are recalled. Actually, only the ones that are useful in control synthesis are recalled. The reader is invited to consult [1, pp. 182-185] and [9] to get a more comprehensive presentation.

[1] Residuation achieves equality in the scalar case.

Property 22.3

$$
\begin{array}{llr}
\textit{Left product} & \textit{Right product} & \\
a(a\backslash x) \preceq x & (x/a)a \preceq x & (22.6)\\
a\backslash(ax) \succeq x & (xa)/a \succeq x & (22.7)\\
a\big(a\backslash(ax)\big) = ax & \big((xa)/a\big)a = xa & (22.8)\\
(ab)\backslash x = b\backslash(a\backslash x) & x/(ba) = (x/a)/b & (22.9)\\
(a\backslash x)b \preceq a\backslash(xb) & b(x/a) \preceq (bx)/a & (22.10)
\end{array}
$$

Proof. Proofs are given for the left product, the same hold for the right one.
Equations (22.6) and (22.7): Theorem 22.1 leads to $L_a \circ L_a^\sharp \preceq \mathsf{Id}_{\mathscr{D}}$ (*i.e.*, $a(a\backslash x) \preceq x$) and $L_a^\sharp \circ L_a \succeq \mathsf{Id}_{\mathscr{D}}$ (*i.e.*, $a\backslash(ax) \succeq x$).
Equation (22.8): Theorem 22.2 leads to $L_a \circ L_a^\sharp \circ L_a = L_a$ (*i.e.*, $a\big(a\backslash(ax)\big) = ax$).
Equation (22.9): It is a direct consequence of the associativity of the product law.
Equation (22.10): According to associativity of the product law, $a(xb) = (ax)b$ then $a\big((a\backslash x)b\big) = \big(a(a\backslash x)\big)b$ and $a\backslash\Big(a\big((a\backslash x)b\big)\Big) = a\backslash\Big(\big(a(a\backslash x)\big)b\Big)$. According to Equation (22.7), $(a\backslash x)b \preceq a\backslash\Big(a\big((a\backslash x)b\big)\Big)$ and according to Equation (22.6) $a\backslash\Big(\big(a(a\backslash x)\big)b\Big) \preceq a\backslash(xb)$. Then, inequality (22.10) holds. □

22.2.2 *Implicit Equations over Dioids*

Definition 22.3. *Let $\mathscr{D}$ be a dioid and $a \in \mathscr{D}$. The Kleene star operator is defined as follows:*

$$a^* \triangleq e \oplus a \oplus a^2 \oplus \ldots = \bigoplus_{i\in\mathbb{N}_0} a^i,$$

with $a^0 = e$ and $a^n = \underbrace{a \otimes a \otimes \cdots \otimes a}_{n\ times}$.

Theorem 22.3. *a^*b is the least solution of equation $x = ax \oplus b$.*

Proof. By repeatedly inserting equation into itself, one can write:

$$x = ax \oplus b = a(ax \oplus b) \oplus b = a^2x \oplus ab \oplus b = a^nx \oplus a^{n-1}b \oplus \cdots \oplus a^2b \oplus ab \oplus b.$$

According to Definition 22.3, this leads to $x = a^\infty x \oplus a^*b$. Hence $x \succeq a^*b$. Furthermore, it can be shown that a^*b is a solution. Indeed:

$$a(a^*b) \oplus b = (aa^* \oplus e)b = a^*b,$$

which implies that a^*b is the smallest solution. □

Now we give some useful properties of the Kleene star operator.

Property 22.4 Let $\mathscr{D}$ be a complete dioid. $\forall a, b \in \mathscr{D}$,

$$a^* a^* = a^* \tag{22.11}$$
$$(a^*)^* = a^* \tag{22.12}$$
$$a(ba)^* = (ab)^* a \tag{22.13}$$
$$(a \oplus b)^* = (a^* b)^* a^* = b^*(ab^*)^* = (a \oplus b)^* a^* = b^*(a \oplus b)^* \tag{22.14}$$

Proof. Equation (22.11): $a^* \otimes a^* = (e \oplus a \oplus a^2 \oplus \ldots) \otimes (e \oplus a \oplus a^2 \oplus \ldots) = e \oplus a \oplus a^2 \oplus a^3 \oplus a^4 \oplus \ldots = a^*$.
Equation (22.12): $(a^*)^* = e \oplus a^* \oplus a^* a^* \oplus \ldots = e \oplus a^* \oplus a^*$.
Equation (22.13): $a(ba)^* = a(e \oplus ba \oplus baba \oplus \ldots) = a \oplus aba \oplus ababa \oplus \ldots = (e \oplus ab \oplus abab \oplus \ldots) \otimes a = (ab)^* a$.
Equation (22.14): $x = (a \oplus b)^*$ is the smallest solution of $x = (a \oplus b) \otimes x \oplus e = ax \oplus bx \oplus e$. On the other hand, this equation admits $x = (a^* b)x \oplus a^* = (a^* b)^* a^*$ as its smallest solution. Hence $(a \oplus b)^* = (a^* b)^* a^*$. The other equalities can be obtained by commuting a and b. $\square$

Theorem 22.4. [7, 32] *Let $\mathscr{D}$ be a complete dioid and $a, b \in \mathscr{D}$. Elements $(a \circ\!\!\!\backslash a)$ and $(b \phi\!\!\!/ b)$ are such that*

$$\begin{aligned} a \circ\!\!\!\backslash a &= (a \circ\!\!\!\backslash a)^*, \\ b \phi\!\!\!/ b &= (b \phi\!\!\!/ b)^*. \end{aligned} \tag{22.15}$$

Proof. According to Equation (22.7), $a \circ\!\!\!\backslash (ae) = a \circ\!\!\!\backslash a \succeq e$. Furthermore, according to Theorem 22.2, $a \circ\!\!\!\backslash (a \otimes (a \circ\!\!\!\backslash a)) = a \circ\!\!\!\backslash a$. Moreover, according to Inequality (22.10), $a \circ\!\!\!\backslash (a \otimes (a \circ\!\!\!\backslash a)) \succeq (a \circ\!\!\!\backslash a) \otimes (a \circ\!\!\!\backslash a)$. Hence, $e \preceq (a \circ\!\!\!\backslash a) \otimes (a \circ\!\!\!\backslash a) = (a \circ\!\!\!\backslash a)^2 \preceq (a \circ\!\!\!\backslash a)$. By recalling that the product law is isotone, these inequalities can be extended to $e \preceq (a \circ\!\!\!\backslash a)^n \preceq (a \circ\!\!\!\backslash a)$. And then by considering the definition of the Kleene star operator, $e \preceq (a \circ\!\!\!\backslash a)^* = (a \circ\!\!\!\backslash a)$. A similar proof can be developed for $b \phi\!\!\!/ b$. $\square$

Theorem 22.5. *Let $\mathscr{D}$ be a complete dioid and $x, a \in \mathscr{D}$. The greatest solution of*

$$x^* \preceq a^* \tag{22.16}$$

is $x = a^*$.

Proof. First, according to the Kleene star definition (see Definition 22.3), the following equivalence holds:

$$x^* = e \oplus x \oplus x^2 \oplus \ldots \preceq a^* \Leftrightarrow \begin{cases} e \preceq a^* \\ x \preceq a^* \\ x^2 \preceq a^* \\ \ldots \\ x^n \preceq a^* \\ \ldots \end{cases} . \tag{22.17}$$

Let us focus on the right-hand side of this equivalence. From Definition 22.3, the first inequality holds. The second admits a^* as its greatest solution, which is also a solution to the other ones. Indeed, according to Equation (22.11), $a^*a^* = a^*$ and $(a^*)^n = a^*$. Hence $x = a^*$ satisfies all these inequalities and is then the greatest solution. □

22.2.3 Dioid $\mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$

In the previous chapter, the semiring of non-decreasing series in two variables with exponents in $\overline{\mathbb{Z}}$ and with Boolean coefficients, denoted $\mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$, was considered to model timed event graphs and to obtain transfer relations between transitions. In this section, the results introduced in the previous chapter are extended. Formally, the considered series are defined as follows:

$$s = \bigoplus_{i \in \mathbb{N}_0} s(i)\gamma^{n_i}\delta^{t_i}, \tag{22.18}$$

with $s(i) \in \{e, \varepsilon\}$ and $n_i, t_i \in \mathbb{Z}$. The support of s is defined by $Supp(s) = \{i \in \mathbb{N}_0 | s(i) \neq \varepsilon\}$. The valuation in γ of s is defined as: $val(s) = min\{n_i | i \in Supp(s)\}$. A series $s \in \mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is said to be a polynomial if $Supp(s)$ is finite. Furthermore, a polynomial is said to be a monomial if there is only one element.

Definition 22.4 (Causality). *A series $s \in \mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is causal if $s = \varepsilon$ or if both $val(s) \geq 0$ and $s \succeq \gamma^{val(s)}$. The set of causal elements of $\mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$ has a complete semiring structure denoted $\mathscr{M}_{in}^{ax+}[\![\gamma,\delta]\!]$.*

Definition 22.5 (Periodicity). *A series $s \in \mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is said to be a periodic series if it can be written as $s = p \oplus q \otimes r^*$, with $p = \bigoplus_{i=1}^{n} \gamma^{n_i}\delta^{t_i}$, $q = \bigoplus_{j=1}^{m} \gamma^{N_j}\delta^{T_j}$ are polynomials and $r = \gamma^{\nu}\delta^{\tau}$, with $\nu, \tau \in \mathbb{N}$, is a monomial depicting the periodicity and allowing to define the asymptotic slope of the series as $\sigma_\infty(s) = \nu/\tau$. A matrix is said to be periodic if all its entries are periodic series.*

Definition 22.6 (Realizability). *A series $s \in \mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is said to be realizable if there exists matrices such that $s = \boldsymbol{C}(\gamma\boldsymbol{A_1} \oplus \delta\boldsymbol{A_2})^*\boldsymbol{B}$ where $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are $n \times n$ matrices with n a finite integer, $\boldsymbol{B}$ and $\boldsymbol{C}$ are $n \times 1$ and $1 \times n$ matrices respectively, with the entries of these matrices are in $\{\varepsilon, e\}$. A matrix $H \in \mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]^{p \times q}$ is said to be realizable if all its entries are realizable.*

In other words, a matrix H is realizable if it corresponds to the transfer relation of a timed event graph[2].

[2] Timed event graphs constitute a subclass of Petri nets as introduced in the previous chapters.

Theorem 22.6. [4] *The following statements are equivalent:*

(a) A series s is realizable.
(b) A series s is periodic and causal.

Definition 22.7 (Canonical form of polynomials). *A polynomial* $p = \bigoplus_{i=0}^{n} \gamma^{n_i}\delta^{t_i}$ *is in canonical form if* $n_0 < n_1 < \cdots < n_n$ *and* $t_0 < t_1 < \cdots < t_n$.

Definition 22.8 (Canonical form of series). *A series* $s = p \oplus q \otimes r^*$, *with* $p = \bigoplus_{i=0}^{n} \gamma^{n_i}\delta^{t_i}$, $q = \bigoplus_{j=0}^{m} \gamma^{N_j}\delta^{T_j}$ *and* $r = \gamma^{\nu}\delta^{\tau}$ *is in* ***proper form***, *if*

- *p and q are in canonical form*
- $n_n < N_0$ *and* $t_n < T_0$
- $N_m - N_0 < \nu$ *and* $T_m - T_0 < \tau$

furthermore, a series is in ***canonical form*** *if it is in proper form and if*

- (n_n, t_n) *is minimal*
- (ν, τ) *is minimal*

Sum, product, Kleene star, and residuation of periodic series are well-defined (see [12, 20]), and algorithms and software toolboxes are available in order to handle periodic series and to compute transfer relations (see [6]). Below, useful computational rules are recalled

$$\delta^{t_1}\gamma^{n} \oplus \delta^{t_2}\gamma^{n} = \delta^{max(t_1,t_2)}\gamma^{n}, \tag{22.19}$$

$$\delta^{t}\gamma^{n_1} \oplus \delta^{t}\gamma^{n_2} = \delta^{t}\gamma^{min(n_1,n_2)}, \tag{22.20}$$

$$\delta^{t_1}\gamma^{n_1} \otimes \delta^{t_2}\gamma^{n_2} = \delta^{t_1+t_2}\gamma^{n_1+n_2}, \tag{22.21}$$

$$(\delta^{t_1}\gamma^{n_1}) \backslash\!\!\circ (\delta^{t_2}\gamma^{n_2}) = \delta^{t_2-t_1}\gamma^{n_2-n_1}, \tag{22.22}$$

$$\delta^{t_1}\gamma^{n_1} \wedge \delta^{t_2}\gamma^{n_2} = \delta^{min(t_1,t_2)}\gamma^{max(n_1,n_2)}. \tag{22.23}$$

Furthermore, we recall that the order relation is such that $\delta^{t_1}\gamma^{n_1} \succeq \delta^{t_2}\gamma^{n_2} \Leftrightarrow n_1 \leq n_2$ and $t_1 \geq t_2$. Let p and p' be two polynomials composed of m and m' monomials respectively, then the following rules hold:

$$p \oplus p' = \bigoplus_{i=1}^{m} \gamma^{n_i}\delta^{t_i} \oplus \bigoplus_{j=1}^{m'} \gamma^{n'_j}\delta^{t'_j}, \tag{22.24}$$

$$p \otimes p' = \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{m'} \gamma^{n_i+n'_j}\delta^{t_i+t'_j}, \tag{22.25}$$

$$p \wedge p' = \bigoplus_{i=1}^{m} (\gamma^{n_i}\delta^{t_i} \wedge p') = \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{m'} \gamma^{max(n_i,n'_j)}\delta^{min(t_i,t'_j)}, \tag{22.26}$$

$$p' \backslash\!\!\circ p = (\bigoplus_{j=1}^{m'} \gamma^{n'_j}\delta^{t'_j}) \backslash\!\!\circ p = \bigwedge_{j=1}^{m'} (\gamma^{-n'_j}\delta^{-t'_j} \otimes p) = \bigwedge_{j=1}^{m'}\bigoplus_{i=1}^{m} \gamma^{n_i-n'_j}\delta^{t_i-t'_j}. \tag{22.27}$$

Let s and s' be two series, the asymptotic slopes satisfy the following rules

$$\sigma_\infty(s \oplus s') = \min\big(\sigma_\infty(s), \sigma_\infty(s')\big), \tag{22.28}$$
$$\sigma_\infty(s \otimes s') = \min\big(\sigma_\infty(s), \sigma_\infty(s')\big), \tag{22.29}$$
$$\sigma_\infty(s \wedge s') = \max\big(\sigma_\infty(s), \sigma_\infty(s')\big), \tag{22.30}$$
$$\text{if } \sigma_\infty(s) \leq \sigma_\infty(s') \text{ then } \sigma_\infty(s' \backslash\!\!\!\circ\, s) = \sigma_\infty(s), \tag{22.31}$$
$$\text{else } s' \backslash\!\!\!\circ\, s = \varepsilon.$$

Theorem 22.7. [7] *The canonical injection* $\mathsf{Id}_{|\mathcal{M}_{in}^{ax+}[\gamma,\delta]} : \mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!] \to \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ *is residuated and its residual is denoted* $\mathsf{Pr}_+ : \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!] \to \mathcal{M}_{in}^{ax+}[\![\gamma,\delta]\!]$.

$\mathsf{Pr}_+(s)$ is the greatest causal series less than or equal to series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. From a practical point of view, for all series $s \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, the computation of $\mathsf{Pr}_+(s)$ is obtained by:

$$\mathsf{Pr}_+\Big(\bigoplus_{i\in I} s(n_i,t_i)\gamma^{n_i}\delta^{t_i}\Big) = \bigoplus_{i\in I} s_+(n_i,t_i)\gamma^{n_i}\delta^{t_i}$$

where

$$s_+(n_i,t_i) = \begin{cases} e & \text{if } (n_i,t_i) \geq (0,0) \\ \varepsilon & \text{otherwise} \end{cases}.$$

22.3 Control

22.3.1 Optimal Open-Loop Control

As in classical linear system theory, the control of systems aims at obtaining control inputs of these systems in order to achieve a behavioral specification. The first result appeared in [4]. The optimal control proposed therein tracks a trajectory *a priori* known in order to minimize the just-in-time criterion. It is an open-loop control strategy, well detailed in [1], and some refinements are proposed later in [29, 30]. The solved problem is the following.

The model of a linear system in a dioid is assumed to be known, which implies that entries of matrices $\boldsymbol{A} \in \mathscr{D}^{n\times n}$, $\boldsymbol{B} \in \mathscr{D}^{n\times p}$, $\boldsymbol{C} \in \mathscr{D}^{q\times n}$ are given and the evolution of the state vector $\boldsymbol{x} \in \mathscr{D}^n$ and the output vector $\boldsymbol{y} \in \mathscr{D}^q$ can be predicted thanks to the model given by:

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned} \tag{22.32}$$

where $\boldsymbol{u} \in \mathscr{D}^p$ is the input vector. Furthermore, a specified output $\boldsymbol{z} \in \mathscr{D}^q$ is assumed to be known. This corresponds to a trajectory to track. It can be shown that there exists a unique greatest input denoted $\boldsymbol{u}_{opt}$ such that the output corresponding to this control, denoted $\boldsymbol{y}_{opt}$, is lower than or equal to the specified output $\boldsymbol{z}$. In practice, the greatest input means that the inputs in the system will occur as late as possible while ensuring that the occurrence dates of the output will be lower than the one given by the specified output $\boldsymbol{z}$. Control $\boldsymbol{u}_{opt}$ is then optimal according to the just-in-time criterion, *i.e.*, the inputs will occur as late as possible while ensuring that the corresponding outputs satisfy the constraints given by $\boldsymbol{z}$. This kind of problem is very popular in the framework of manufacturing systems: the specified output $\boldsymbol{z}$ corresponds to the customer demand, and the optimal control $\boldsymbol{u}_{opt}$ corresponds to the input of raw parts in the manufacturing system. The latter is delayed as much as possible, while ensuring that optimal output $\boldsymbol{y}_{opt}$ of the processed parts occurs before the customer demand. Hence the internal stocks are reduced as much as possible, which is worthwhile from an economic point of view. According to Theorem 22.3, the smallest solution of System (22.32) is given by:

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u}. \end{aligned} \tag{22.33}$$

This smallest solution represents the fastest evolution of the system. Formally, the optimal problem consists in computing the greatest $\boldsymbol{u}$ such that $\boldsymbol{y} \preceq \boldsymbol{z}$. By recalling that the left product is residuated (see Example 22.1), the following equivalence holds:

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u} \preceq \boldsymbol{z} \Leftrightarrow \boldsymbol{u} \preceq (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \circ\!\!\!\!\backslash\, \boldsymbol{z}.$$

In other words the greatest control achieving the objective is:

$$\boldsymbol{u}_{opt} = (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \circ\!\!\!\!\backslash\, \boldsymbol{z}. \tag{22.34}$$

22.3.1.1 Illustration

High-Throughput Screening (HTS) systems allow researchers to quickly conduct millions of chemical, genetic, or pharmacological tests. In the previous chapter, an elementary one was considered as an illustrative example. Below, its model in the semiring $\mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is recalled.

$$\begin{aligned}
\boldsymbol{x}(\gamma,\delta) &= \begin{bmatrix}
\varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\delta^7 & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma^2 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & e & \varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^{12} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^5 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta & \varepsilon
\end{bmatrix} \boldsymbol{x}(\gamma,\delta) \oplus \begin{bmatrix}
e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & e & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & e \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon
\end{bmatrix} \boldsymbol{u}(\gamma,\delta) \\
y(\gamma,\delta) &= \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \boldsymbol{x}(\gamma,\delta)
\end{aligned} \tag{22.35}$$

This model leads to the following transfer matrices, where the entries are periodic series in $\mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$:

$$
\begin{aligned}
\boldsymbol{x}(\gamma,\delta) &= \begin{bmatrix}
(e)(\gamma\delta^8)^* & (\gamma\delta^2)(\gamma\delta^8)^* & (\gamma^3\delta^{19})(\gamma\delta^8)^* \\
(\delta^7)(\gamma\delta^8)^* & (\delta)(\gamma\delta^8)^* & (\gamma^2\delta^{18})(\gamma\delta^8)^* \\
(\delta^8)(\gamma\delta^8)^* & (\delta^2)(\gamma\delta^8)^* & (\gamma^2\delta^{19})(\gamma\delta^8)^* \\
(\gamma^2\delta^{20})(\gamma\delta^8)^* & e\oplus(\gamma^2\delta^{14})(\gamma\delta^8)^* & \gamma^2\delta^{17}\oplus(\gamma^3\delta^{23})(\gamma\delta^8)^* \\
(\delta^7)(\gamma\delta^8)^* & (\delta)(\gamma\delta^8)^* & (\gamma^2\delta^{18})(\gamma\delta^8)^* \\
(\delta^8)(\gamma\delta^8)^* & (\delta^2)(\gamma\delta^8)^* & \delta^5\oplus(\gamma\delta^{11})(\gamma\delta^8)^* \\
(\delta^{20})(\gamma\delta^8)^* & (\delta^{14})(\gamma\delta^8)^* & \delta^{17}\oplus(\gamma\delta^{23})(\gamma\delta^8)^* \\
\varepsilon & \varepsilon & (e)(\gamma\delta^6)^* \\
\varepsilon & \varepsilon & (\delta^5)(\gamma\delta^6)^* \\
\varepsilon & \varepsilon & (\delta^6)(\gamma\delta^6)^*
\end{bmatrix} \boldsymbol{u}(\gamma,\delta) \\
y(\gamma,\delta) &= \begin{bmatrix}(\delta^{20})(\gamma\delta^8)^* & (\delta^{14})(\gamma\delta^8)^* & \delta^{17}\oplus(\gamma\delta^{23})(\gamma\delta^8)^*\end{bmatrix}\boldsymbol{u}(\gamma,\delta).
\end{aligned}
\tag{22.36}
$$

The following reference trajectory is assumed to be known:

$$z(\gamma,\delta) = \delta^{37}\oplus\gamma\delta^{42}\oplus\gamma^3\delta^{55}\oplus\gamma^4\delta^{+\infty}. \tag{22.37}$$

It models that one part is expected before or at time 37, 2 parts before (or at) time 42 and 4 parts before (or at) time 55. It must be noted that the numbering of parts starts at 0[3]. According to Equation (22.34) and the computation rules given in the previous section, the optimal firing trajectories of the three inputs are as follows:

$$\boldsymbol{u}_{opt}(\gamma,\delta) = \begin{bmatrix}
\delta^6\oplus\gamma\delta^{14}\oplus\gamma^2\delta^{22}\oplus\gamma^3\delta^{35}\oplus\gamma^4\delta^{+\infty} \\
\delta^{12}\oplus\gamma\delta^{20}\oplus\gamma^2\delta^{28}\oplus\gamma^3\delta^{41}\oplus\gamma^4\delta^{+\infty} \\
\delta^{11}\oplus\gamma\delta^{19}\oplus\gamma^2\delta^{25}\oplus\gamma^3\delta^{38}\oplus\gamma^4\delta^{+\infty}
\end{bmatrix}, \tag{22.38}$$

and the resulting optimal output is:

$$y_{opt}(\gamma,\delta) = \boldsymbol{CA}^*\boldsymbol{B}\boldsymbol{u}_{opt}(\gamma,\delta) = \delta^{28}\oplus\gamma\delta^{36}\oplus\gamma^2\delta^{42}\oplus\gamma^3\delta^{55}\oplus\gamma^4\delta^{+\infty}, \tag{22.39}$$

which means that the first part will exit the system at time 28, the second part at time 36, the third part at time 42 and the fourth part at time 55. This output trajectory is the greatest in the image of matrix $\boldsymbol{CA}^*\boldsymbol{B}$, that is the greatest reachable output, such that the events occur before the required dates. This example[4] was computed with library MinMaxGD (a C++ library which is interfaced as a toolbox for both Scilab and Matlab software, see [6]).

---

[3] Following the convention defined between remarks 5.22 and 5.23 in [1, Section 5.4].

[4] The sources are available at the following URL: `www.istia.univ-angers.fr/~hardouin/DISCBookChapterControlInDioid.html`.

22.3.2 Optimal Input Filtering in Dioids

In the previous section, the proposed control law leads to the optimal tracking of a trajectory, which is assumed to be *a priori* known. In this section, the aim is to track a reference model in an optimal manner. This means that the specified output is a priori unknown and the goal is to control the system so that it behaves as the reference model which was built from a specification. Hence, this problem is a model matching problem. The reference model describing the specified behavior is assumed to be a linear model in the considered dioid. The objective is to synthesize an optimal filter controlling the inputs (sometimes called a precompensator in the related literature) in order to get an output as close as possible to the one you would obtain by applying this input to the reference model.

Formally, the specified behavior is assumed to be described by a transfer matrix denoted $\boldsymbol{G} \in \mathscr{D}^{q\times m}$ leading to the outputs $\boldsymbol{z} = \boldsymbol{G}\boldsymbol{v} \in \mathscr{D}^{q}$ where $\boldsymbol{v} \in \mathscr{D}^{m}$ is the model input. The system is assumed to be described by the transfer relation $\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B} \in \mathscr{D}^{q\times p}$ and its output $\boldsymbol{y}$ is given by $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u} \in \mathscr{D}^{q}$. The control is assumed to be given by a filter $\boldsymbol{P} \in \mathscr{D}^{p\times m}$, *i.e.*, $\boldsymbol{u} = \boldsymbol{P}\boldsymbol{v} \in \mathscr{D}^{p}$. Therefore the aim is to synthesize this filter such that, for all $\boldsymbol{v}$:

$$\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{P}\boldsymbol{v} \preceq \boldsymbol{G}\boldsymbol{v}, \tag{22.40}$$

which is equivalent to

$$\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{P} \preceq \boldsymbol{G}. \tag{22.41}$$

Thanks to residuation theory, the following equivalence holds:

$$\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{P} \preceq \boldsymbol{G} \Leftrightarrow \boldsymbol{P} \preceq (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \circ\!\!\!\!\backslash\, \boldsymbol{G} \tag{22.42}$$

Hence the optimal filter is given by $\boldsymbol{P}_{opt} = (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \circ\!\!\!\!\backslash\, \boldsymbol{G}$ and it leads to the control $\boldsymbol{u}_{opt} = \boldsymbol{P}_{opt}\boldsymbol{v}$. Furthermore, if $\boldsymbol{G}$ is in the image of the transfer matrix $\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}$ (*i.e.*, $\exists \boldsymbol{L}$ such that $\boldsymbol{G} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{L}$), then the following equality holds:

$$\boldsymbol{G} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{P}_{opt}. \tag{22.43}$$

Obviously, this reference model is reachable since it is sufficient to take a filter equal to the identity matrix to satisfy the equality. The optimal filter is given by $\boldsymbol{P}_{opt} = (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \circ\!\!\!\!\backslash\, (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})$ and Equality (22.43) is satisfied. Formally $\boldsymbol{G} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{P}_{opt} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\big((\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \circ\!\!\!\!\backslash\, (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})\big)$. This means that output $\boldsymbol{y}$ will be unchanged by the optimal filter, *i.e.*, $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{v} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{P}\boldsymbol{v}$ hence the output occurrences of the controlled system will be as fast as the one of the system without control. Nevertheless the control $\boldsymbol{u}_{opt} = \boldsymbol{P}_{opt}\boldsymbol{v}$ will be the greatest one leading to this unchanged output. In the framework of manufacturing systems, this means that the job will start as late as possible, while preserving the output. Hence the work-in-progress will be reduced as much as possible. This kind of controller is called neutral due to its neutral action on the output behavior. The benefit lies on a reduction of the internal stocks.

22.3.2.1 Illustration

By applying these results to the previous example, the optimal input filter is given by:

$$\boldsymbol{P}_{opt}(\gamma,\delta) = (\boldsymbol{CA}^*\boldsymbol{B}) \backslash\!\!\circ (\boldsymbol{CA}^*\boldsymbol{B}) = \begin{bmatrix} (\gamma\delta^8)^* & (\delta^{-6})(\gamma\delta^8)^* & (\delta^{-5})(\gamma\delta^8)^* \\ (\delta^6)(\gamma\delta^8)^* & (\gamma\delta^8)^* & (\delta^1)(\gamma\delta^8)^* \\ (\delta^3)(\gamma\delta^8)^* & (\delta^{-3})(\gamma\delta^8)^* & e \oplus (\gamma\delta^6)(\gamma\delta^8)^* \end{bmatrix} \tag{22.44}$$

It must be noted that some powers are negative (see the computational rule 22.27). This means that the filter is not causal, and therefore not realizable (see Theorem 22.6). In practice, it is necessary to project it in the semiring of causal series, denoted $\mathscr{M}_{in}^{ax+}[\![\gamma,\delta]\!]$. This is done thanks to the rule introduced in Theorem 22.7.

By applying this projector to the previous filter, the optimal causal filter is given by:

$$\boldsymbol{P}_{opt}^{+}(\gamma,\delta) = \mathsf{Pr}_{+}(\boldsymbol{P}_{opt}) = \begin{bmatrix} (\gamma\delta^8)^* & (\gamma\delta^2)(\gamma\delta^8)^* & (\gamma\delta^3)(\gamma\delta^8)^* \\ (\delta^6)(\gamma\delta^8)^* & (\gamma\delta^8)^* & (\delta^1)(\gamma\delta^8)^* \\ (\delta^3)(\gamma\delta^8)^* & (\gamma\delta^5)(\gamma\delta^8)^* & e \oplus (\gamma\delta^6)(\gamma\delta^8)^* \end{bmatrix}. \tag{22.45}$$

This filter describes a 3-input 3-output system of which a realization can be obtained either in time or in event domain. As explained in the previous chapter, in the time domain, the variables, and their associated trajectory, will represent the maximal number of events occurred at a time t. Dually, in the event domain, they will represent the earliest occurrence dates of the k^{th} event. The adopted point of view will depend on the technological context leading to the implementation. Indeed, the control law can be implemented in a control command system or in a PLC, which can be either event driven or synchronized with a clock. In both cases, the following method may be used to obtain a realization of the control law. First, we recall the expression of the control law $u_i = \bigoplus_{j=1}^{p} (\boldsymbol{P}_{opt})_{ij} v_j$ with $i \in [1..p]$, where each entry $(\boldsymbol{P}_{opt})_{ij}$ is a periodic series which can be written as follows: $(\boldsymbol{P}_{opt})_{ij} = p_{ij} \oplus q_{ij} r_{ij}^*$. This leads to the following control law:

$$u_i = \bigoplus_{j=1}^{p} (\boldsymbol{P}_{opt})_{ij} v_j \tag{22.46}$$

with

$$(\boldsymbol{P}_{opt})_{ij} v_j = p_{ij} v_j \oplus q_{ij} r_{ij}^* v_j.$$

The last line can be written in an explicit form, by introducing an internal variable ζ_{ij} (we recall that $x = a^*b$ is the least solution of $x = ax \oplus b$, see Theorem 22.3):

$$\begin{aligned} \zeta_{ij} &= r_{ij}\zeta_{ij} \oplus q_{ij} v_j \\ (\boldsymbol{P}_{opt})_{ij} v_j &= p_{ij} v_j \oplus \zeta_{ij} \end{aligned} . \tag{22.47}$$

This explicit formulation may be used to obtain the control law, either in the time domain or in the event domain. Below, the control law (Equation (22.45)) is given in the time domain:

$$\begin{aligned}
\zeta_{11}(t) &= min(1+\zeta_{11}(t-8), v_1(t)) \\
\zeta_{12}(t) &= min(1+\zeta_{12}(t-8), 1+v_2(t-2)) \\
\zeta_{13}(t) &= min(1+\zeta_{13}(t-8), 1+v_3(t-3)) \\
\zeta_{21}(t) &= min(1+\zeta_{21}(t-8), v_1(t-6)) \\
\zeta_{22}(t) &= min(1+\zeta_{22}(t-8), v_2(t)) \\
\zeta_{23}(t) &= min(1+\zeta_{23}(t-8), v_3(t-1)) \\
\zeta_{31}(t) &= min(1+\zeta_{31}(t-8), v_1(t-3)) \\
\zeta_{32}(t) &= min(1+\zeta_{32}(t-8), 1+v_2(t-5)) \\
\zeta_{33}(t) &= min(1+\zeta_{33}(t-8), 1+v_3(t-6)) \\
u_1(t) &= min(\zeta_{11}(t), \zeta_{12}(t), \zeta_{13}(t)) \\
u_2(t) &= min(\zeta_{21}(t), \zeta_{22}(t), \zeta_{23}(t)) \\
u_3(t) &= min(\zeta_{21}(t), \zeta_{22}(t), \zeta_{23}(t), v_3(t)).
\end{aligned} \tag{22.48}$$

22.3.3 Closed-Loop Control in Dioids

The two previous sections have proposed open-loop control strategies. In this section, the measurement of the system outputs are taken into account in order to compute the control inputs. The closed-loop control strategy considered is given in Fig. 22.1. It aims at modifying the dynamics of system $\boldsymbol{CA}^*\boldsymbol{B} \in \mathscr{D}^{q\times p}$ by using a controller $\boldsymbol{F} \in \mathscr{D}^{p\times q}$ located between the output $\boldsymbol{y} \in \mathscr{D}^q$ and the input $\boldsymbol{u} \in \mathscr{D}^p$ of the system and a filter $\boldsymbol{P} \in \mathscr{D}^{p\times m}$ located upstream of the system, as the one considered in the previous section. The controllers are chosen in order to obtain a controlled system as close as possible to a given reference model $\boldsymbol{G} \in \mathscr{D}^{q\times m}$. The latter is assumed to be a linear model in the considered dioid.

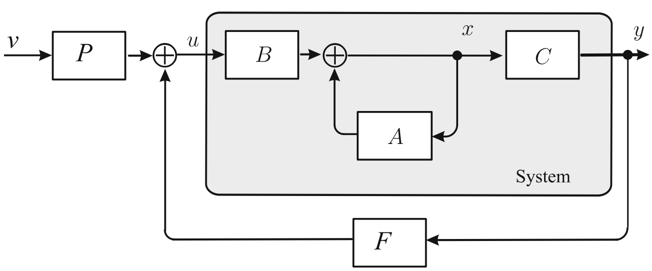

Fig. 22.1 Control architecture, the system is controlled by output feedback $\boldsymbol{F}$ and filter $\boldsymbol{P}$

Formally, the control input is equal to $\boldsymbol{u} = \boldsymbol{F}\boldsymbol{y} \oplus \boldsymbol{P}\boldsymbol{v} \in \mathscr{D}^p$ and leads to the following closed-loop behavior:

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{B}(\boldsymbol{F}\boldsymbol{y} \oplus \boldsymbol{P}\boldsymbol{v}) \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \preceq \boldsymbol{G}\boldsymbol{v} \qquad \forall \boldsymbol{v}. \end{aligned} \tag{22.49}$$

By replacing $\boldsymbol{y}$ in the first equation, one can write

$$\begin{aligned} \boldsymbol{x} &= (\boldsymbol{A} \oplus \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})\boldsymbol{x} \oplus \boldsymbol{B}\boldsymbol{P}\boldsymbol{v} = (\boldsymbol{A} \oplus \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})^*\boldsymbol{B}\boldsymbol{P}\boldsymbol{v} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} = \boldsymbol{C}(\boldsymbol{A} \oplus \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})^*\boldsymbol{B}\boldsymbol{P}\boldsymbol{v} \preceq \boldsymbol{G}\boldsymbol{v} \qquad \forall \boldsymbol{v}. \end{aligned} \tag{22.50}$$

Hence the problem is to find controllers $\boldsymbol{F}$ and $\boldsymbol{P}$ such that:

$$\boldsymbol{C}(\boldsymbol{A} \oplus \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})^*\boldsymbol{B}\boldsymbol{P} \preceq \boldsymbol{G}. \tag{22.51}$$

By recalling that $(a \oplus b)^* = a^*(ba^*)^* = (a^*b)^*a^*$ (see Equation (22.14)), this equation can be written as:

$$\boldsymbol{C}\boldsymbol{A}^*(\boldsymbol{B}\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*)^*\boldsymbol{B}\boldsymbol{P} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}(\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})^*\boldsymbol{P} \preceq \boldsymbol{G}. \tag{22.52}$$

According to the Kleene star definition $(\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})^* = \boldsymbol{E} \oplus \boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B} \oplus (\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})^2 \oplus \ldots$, so Equation (22.52) implies the following constraint on the filter $\boldsymbol{P}$:

$$\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{P} \preceq \boldsymbol{G}, \tag{22.53}$$

which is equivalent to:

$$\boldsymbol{P} \preceq (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \backslash\!\!\circ\, \boldsymbol{G} = \boldsymbol{P}_{opt}, \tag{22.54}$$

where $\boldsymbol{P}_{opt}$ is the same as the optimal filter introduced in Section 22.3.2. By considering this optimal solution in Equation (22.52), the following equivalences hold:

$$\begin{aligned} \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}(\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})^*\boldsymbol{P}_{opt} \preceq \boldsymbol{G} &\Leftrightarrow \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}(\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})^* \preceq \boldsymbol{G} \,/\!\!\circ\, \boldsymbol{P}_{opt} \\ &\Leftrightarrow (\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})^* \preceq (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \backslash\!\!\circ\, \boldsymbol{G} \,/\!\!\circ\, \boldsymbol{P}_{opt} = \boldsymbol{P}_{opt} \,/\!\!\circ\, \boldsymbol{P}_{opt}. \end{aligned} \tag{22.55}$$

By recalling that $(a \,/\!\!\circ\, a) = (a \,/\!\!\circ\, a)^*$ (see Theorem 22.4) and that $x^* \preceq a^*$ admits $x = a^*$ as its greatest solution (see Theorem 22.5), this equation can be written as:

$$(\boldsymbol{F}\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) \preceq \boldsymbol{P}_{opt} \,/\!\!\circ\, \boldsymbol{P}_{opt}. \tag{22.56}$$

By using the right division, the optimal feedback controller is obtained as follows:

$$\boldsymbol{F} \preceq (\boldsymbol{P}_{opt} \,/\!\!\circ\, \boldsymbol{P}_{opt}) \,/\!\!\circ\, (\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}) = \boldsymbol{P}_{opt} \,/\!\!\circ\, \big((\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B})\boldsymbol{P}_{opt}\big) = \boldsymbol{F}_{opt}. \tag{22.57}$$

And the causal feedback is given by: $\boldsymbol{F}^+_{opt} = \mathsf{Pr}_+(\boldsymbol{F}_{opt})$. It can be noted that, as for the optimal filter, the reference model describes the required behavior. From a practical point of view, an interesting choice is $\boldsymbol{G} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}$. This means that the objective is to obtain the greatest control inputs, while preserving the output of the system without control, *i.e.*, to delay the input dates as much as possible while preserving the output dates.

Remark 22.5. *This kind of closed-loop control can be useful to deal with stabilization problem. This problem is not addressed here, but the reader is invited to consult [13, 8, 24]. In a few words we can recall that a timed event graph, which always admits a max-plus linear model, is stable if and only if the number of tokens in all places is bounded. A sufficient condition is fulfilled if the timed event graph is strongly connected. Hence, it is sufficient to modify the timed event graph thanks to a feedback controller in order to obtain a new graph with only one strongly connected component. Indeed, a feedback controller adds some arcs binding the output transitions and the input transitions. In [13], the author provides a static feedback ($\boldsymbol{F}_s$) allowing the graph to be strongly connected. A static feedback means that the places added between output transitions and input transitions are without delay and contain only tokens. In fact, the author gives the minimal number of tokens which are necessary to preserve the throughput of the system. He has also shown that this minimal number of tokens (which can be seen as a minimal number of resources) can be obtained by considering an integer linear programming problem. In [8, 24], the feedback is improved by considering the residuation theory and the computation of an optimal control. The idea is to use the static feedback as obtained thanks to the approach in [13] in order to compute a specification $\boldsymbol{G} = \boldsymbol{C}(\boldsymbol{A} \oplus \boldsymbol{B}\boldsymbol{F}_s\boldsymbol{C})\boldsymbol{B}$. Then this specification is considered to find a feedback controller, like the one proposed in this section, which can be seen as a dynamic feedback. It must be noted that the numbers of tokens added in the feedback arcs are the same as the ones obtained by S. Gaubert in [13]. Hence, the minimal number of resources is preserved and the dynamic feedback will only modify the dynamic behavior by adding delay in order to be optimal according to the just-in-time criterion. To summarize, this strategy ensures the stability and is optimal according to the just-in-time criterion. This kind of control leads to the greatest reduction of internal stocks in the context of manufacturing systems, as well as waiting times in a transportation network, and also avoids useless buffer saturation in a computer network.*

22.3.3.1 Illustration

The example of Section 22.3.2.1 is continued, *i.e.*, the reference model is $\boldsymbol{G} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}$. Then, the optimal filter is given by Equation (22.45) and the feedback controller is obtained thanks to Equation (22.57). The practical computation yields:

$$\boldsymbol{F}^+_{opt} = \begin{bmatrix} (\gamma^3\delta^2)\,(\gamma\delta^8)^* \\ (\gamma^2)\,(\gamma\delta^8)^* \\ (\gamma^3\delta^5)\,(\gamma\delta^8)^* \end{bmatrix} \tag{22.58}$$

Then the control law, $\boldsymbol{u} = \boldsymbol{P}^+_{opt}\boldsymbol{v} \oplus \boldsymbol{F}^+_{opt}y$ is obtained by adding the feedback control to law (22.48). By using the same methodology as in Section 22.3.2.1, the control law in the time domain can be obtained. An intermediate variable $\boldsymbol{\beta}$ is considered:

$$\begin{array}{l} \beta_{11}(t) = min(1+\beta_{11}(t-8), 3+y(t-2)) \\ \beta_{21}(t) = min(1+\beta_{21}(t-8), 2+y(t)) \\ \beta_{31}(t) = min(1+\beta_{31}(t-8), 3+y(t-5)), \end{array} \tag{22.59}$$

then the following control law is obtained:

$$\begin{array}{l} u_1(t) = min(\zeta_{11}(t), \zeta_{12}(t), \zeta_{13}(t), \beta_{11}(t)) \\ u_2(t) = min(\zeta_{21}(t), \zeta_{22}(t), \zeta_{23}(t), \beta_{11}(t)) \\ u_3(t) = min(\zeta_{21}(t), \zeta_{22}(t), \zeta_{23}(t), v_3(t), \beta_{11}(t)), \end{array} \tag{22.60}$$

where variable $\boldsymbol{\zeta}$ is given in Equation (22.48) and $\boldsymbol{\beta}$ in Equation (22.59). This equation can easily be implemented in a control system. It can also be realized as a timed event graph, such as the one given in Fig. 22.2.

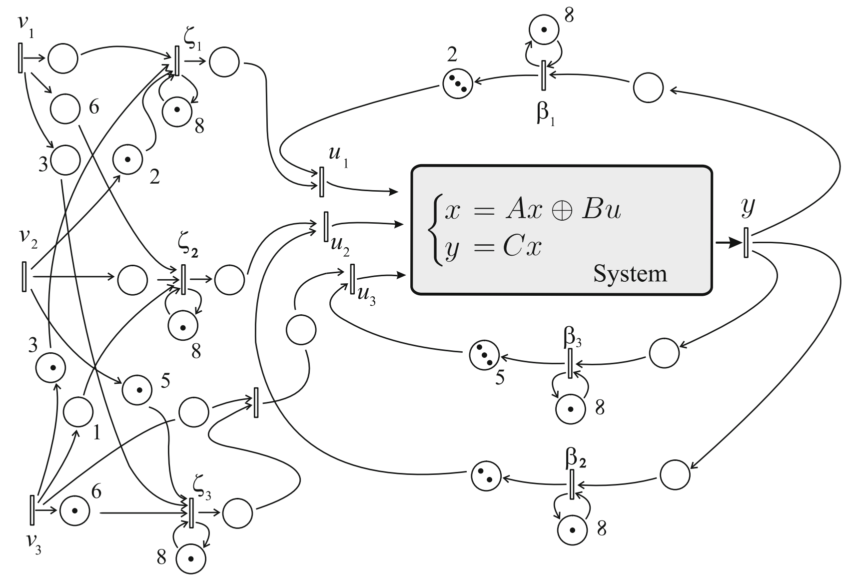

Fig. 22.2 Realization of the optimal filter $\boldsymbol{P}^+_{opt}$ and of the optimal feedback $\boldsymbol{F}^+_{opt}$

22.4 Further Reading

One should notice that some other classical standard problems can be considered from this point of view. Indeed, it is possible to synthesize a linear observer in Luenberger's sense, which allows us to estimate the unmeasured state of the system by considering its output measurement (see [17, 16]). Another related problem consists in the disturbance decoupling problem (see [18, 21]), which aims at taking into account disturbances acting on the system to compute the control law. About uncertainty, the reader is invited to consider a dioid of intervals [15, 25] allowing the synthesis of robust controllers in a bounded error context (see [22, 23]). The identification of model parameters was also considered in some previous work (see e.g., [11, 27, 33]). It can also be noted that some other points of view were considered in the related literature. In [10, 31], the authors considered the so-called model predictive control in order to minimize a quadratic criterion by using a linear programming approach. The same authors have extended their control strategy to a stochastic context (see [34]), in order to take uncertain systems into account.

To summarize, automatic control of discrete-event systems in a dioid framework is still an improving topic with many problems yet to solve.

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