

# Controllability of Second Order Leader-Follower Systems<sup>\*</sup>

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**Abstract:** In this paper we study formation control of a network of communicating mobile agents with double integrator dynamics. All agents run a consensus algorithm, several leader agents are further subject to an external input. We put this problem in a control theoretical framework and show that the controllability of the entire system via external control input depends only on the controllability of the follower system through the leaders. We then show necessary and sufficient conditions for the follower system to be controllable and relate the obtained conditions to the existing results on topological properties of the follower communication graph.

*Keywords:* Networked control systems, leader-follower networks, leader election, controllability, formation control

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## 1. INTRODUCTION

In a network of communicating mobile agents, it is often desirable that these agents autonomously reach an agreement upon a state variable. The achieved value is called consensus value and the network is said to achieve consensus. During the last decade the problem of finding consensus in networks and its applications have received a lot of attention. The research is motivated by a wide variety of applications, such as formation control of unmanned air and underwater vehicles, mobile robots, satellites, sensor networks and inspired by flocking of birds. The consensus mechanisms for networks of identical agents are well-known. For the single integrator consensus, that is, the case where all agents have single integrator dynamics, many consensus results and applications can be found in Ren and Beard (2007), Olfati-Saber et al. (2007), Tanner et al. (2007) and the references therein. Consensus algorithms for agents with higher order integrator dynamics have been studied for example in Ren and Atkins (2005) and Ren et al. (2006).

Usually the desired consensus value is assumed to be the average of the initial states of all agents. However, in some applications it is preferable that the system converges to a given reference value that is not available to all agents. This value can be enforced on the system via one or several leader agents controlled externally. Tanner (2004) has introduced the leader-follower algorithm for single integrator consensus networks and derived controllability criteria for the single leader case. Rahmani et al. (2009) and Ji et al. (2009) have related these criteria to topological properties of the communication graph. This work has been continued e. g. by Borsche and Attia (2010), where the authors also show a statistical approach for choosing the optimal

leader agent. Björkenstam et al. (2006) have suggested a controller design for leader-follower systems. Ren and Sorensen (2006) have presented a control architecture for large scale leader-follower systems.

So far the research has focused on agents with single integrator dynamics. In this contribution we would like to extend the notion of leader-follower system to agents with double integrator dynamics. Our work is motivated by the fact that many real-life applications cannot be modelled as single integrator systems. For example, some mobile robots can be feedback linearised and then described as having double integrator dynamics, which naturally leads to an extended algorithm that enforces consensus on both position and velocity of the agents.

We assume that the leader agents receive both information from the consensus algorithm and an additional control input. We use the consensus algorithm introduced in Ren and Atkins (2005) and show that the consensus value can be controlled through external input under the condition that the followers are controllable by the leaders. We then derive controllability conditions for the follower network with double integrator dynamics and relate them to the topological properties of the graph. We show that these conditions depend on the form of the communication topology.

This paper is organized as follows: In Section 2 we introduce our notation, the system model and the consensus algorithm used, as well as some basic notions on graph theory. In Section 3 we formalise the notion of a leader-follower-network for the double integrator consensus algorithm. Section 4 contains the main result: In Section 4.1 we show that a necessary and sufficient condition for the network to be controllable is that the followers are controllable by the leaders. In Section 4.2 we then derive

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conditions for the controllability of the follower network and relate them to the existing results for leader-follower networks with single integrator dynamics.

## 2. PRELIMINARIES

In this section we introduce our notation and the system model. In order to make this paper self-contained we further present some existing definitions and results in algebraic graph theory and consensus. The results provided in Sections 2.3 and 2.4 are taken from the respective literature.

### 2.1 Notation

Throughout this paper we write  $I^{m \times m}$  for the  $m \times m$  identity matrix and  $1^{k \times m}$  and  $0^{k \times m}$  for the one and zero matrix of size  $k \times m$ , respectively. If the dimension is clear from the context, we simply write  $I$ ,  $1$ ,  $0$ . We write lowercase latin letters for vectors. We write uppercase letters for matrices. If  $A$  is a block matrix, then  $a_{ij}$  is the  $(i, j)$ th entry of  $A$  and  $A_{i,j}$  is the  $(i, j)$ th block (matrix) of  $A$ . If we need to address the whole  $j$ th block row (column) of a matrix, we use the notation  $A_{j,:}$  ( $A_{:,j}$ ). By writing  $A^j$  we denote  $A$  to the power of  $j$ . We reserve  $n$  for the number of agents in the formation,  $k$  for the number of follower agents and  $m$  for the number of leaders. We write  $\text{rank}(A)$  for the rank of matrix  $A$ .

### 2.2 Modelling

In this contribution we consider a group of  $n$  mobile agents moving in a two- or three-dimensional space. We assume that the individual agent's dynamics is decoupled along the different dimensions, i. e. that consensus in each direction can be investigated as a one-dimensional problem.

We denote the position of an agent as  $x^i$ ,  $i \in \{1 \dots n\}$ , and its velocity as  $\dot{x}^i$ . The positions (velocities) of all agents are collected in the vector  $x$  ( $\dot{x}$ ). The agents have double integrator dynamics, i. e.  $\ddot{x} = u$ , where  $x, u \in \mathbb{R}^n$ . The agents move in a common reference frame and can communicate their positions and velocities along the same communication topology.

### 2.3 Algebraic Graph Theory

The high-level properties of a network can be modelled by a communication graph  $G = (V, E)$ . For further details on graph theory see e. g. Godsil and Royle (2001). The set of nodes is given by  $V = \{v_1, v_2, \dots, v_n\}$  and corresponds to the different agents and the set of edges  $E \subset V \times V$  represents the communication links between the units. An edge  $\varepsilon_{ij}$  in  $E$  between two nodes signifies that  $v_i$  can transmit information to  $v_j$ . A graph is called undirected if  $\varepsilon_{ij} \in E \Leftrightarrow \varepsilon_{ji} \in E$ , otherwise it is called directed. We further require that the graphs do not have self-loops, i. e. that there is no edge  $\varepsilon_{ii}$ . The union of graphs  $G = G_1 \cup G_2$  is defined for graphs  $G_1, G_2$  on the same set of nodes as  $G = (V, E_1 \cup E_2)$ . A path between two nodes is a set of edges that connects these two nodes. A graph is connected if there is a path between every two nodes. If the graph is disconnected, then it has several connected components.

A graph can be expressed in terms of an adjacency matrix  $A(G)$ , with the entries  $a_{ij} = 1$  if an edge from  $v_j$  to  $v_i$  exists and  $a_{ij} = 0$  otherwise. The degree of a node is given by  $d(v_i) = \sum_j a_{ij}$ . Let  $D(G) = \text{diag}(d(v_i))$  be the degree matrix. Then the graph Laplacian  $L(G)$  is given as

$$L(G) = D(G) - A(G). \quad (1)$$

When clear, we will write  $L$  instead of  $L(G)$ . If a graph is undirected,  $L(G)$  and  $A(G)$  are symmetric.

### 2.4 Consensus Algorithm

A one-dimensional consensus algorithm for agents with double integrator dynamics is given by

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L & -L \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \quad (2)$$

where  $L$  is the Laplacian of the communication graph of  $n$  agents and  $I$  and  $0$  are the  $n \times n$  identity and zero matrices. As mentioned in Section 2.2, we assume that our system is decoupled along the different dimensions and consider the one-dimensional case.

This algorithm is very intuitive and was suggested in Ren and Atkins (2005) for agents with double integrator dynamics. We say that the algorithm achieves consensus asymptotically if as  $t \rightarrow \infty$   $|x^i - x^j| \rightarrow 0$  and  $|\dot{x}^i - \dot{x}^j| \rightarrow 0$  for all  $i, j \in \{1, \dots, n\}$ . Ren and Atkins (2005) show that for  $m \leq 2$ , (2) always achieves consensus if  $L$  is the Laplacian of a connected undirected graph.

We can now define the consensus based leader-follower system.

## 3. HIGHER ORDER LEADER-FOLLOWER SYSTEM MODEL

In this section we are considering a network of  $n$  agents with double integrator dynamics running the consensus algorithm (2). However, we now assume that there are two different kinds of agents in the network: some that strictly follow the consensus protocol and some that can be controlled externally. We call the agents followers and leaders, respectively. For the single integrator consensus this approach has first been introduced by Tanner (2004). As a first step, we would like to extend his approach to higher order systems.

Let there be  $k$  follower and  $m$  leader agents in the network, such that  $m + k = n$ , denoted by the indices  $f$  and  $l$ . Then the graph Laplacian can be partitioned as

$$L = \begin{pmatrix} L_f & l_{lf} \\ l_{fl} & L_l \end{pmatrix} \quad (3)$$

Here  $L_f$  and  $L_l$  are  $k \times k$  and  $m \times m$  symmetric matrices, however they generally no longer have Laplacian matrix properties. The matrix  $l_{lf}$  denotes the information flow from leaders to followers, and the matrix  $l_{fl}$  the information flow from followers to leaders.

Analogously we can rewrite the state vector as

$$x = (x_f, \dot{x}_f, x_l, \dot{x}_l)^T, \quad (4)$$

where  $x_f, \dot{x}_f \in \mathbb{R}^{k \times 1}$ ,  $x_l, \dot{x}_l \in \mathbb{R}^{m \times 1}$ . With this and the partitioning (3) we can rewrite (2) as the following control system

$$\dot{x} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} x + \begin{pmatrix} 0 \\ F \end{pmatrix} v \quad (5)$$

where

$$A = \begin{pmatrix} 0 & I \\ -L_f & -L_f \end{pmatrix}, \quad (6)$$

$$B = \begin{pmatrix} 0 & 0 \\ -l_{lf} & -l_{lf} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ -l_{fl} & -l_{fl} \end{pmatrix}, \quad (7)$$

$$D = \begin{pmatrix} 0 & I \\ -L_l & -L_l \end{pmatrix}, \quad (8)$$

$$F = \begin{pmatrix} 0^{m \times m} \\ I^{m \times m} \end{pmatrix}. \quad (9)$$

Note that  $A \in \mathbb{R}^{2k \times 2k}$ ,  $B \in \mathbb{R}^{2k \times 2m}$ ,  $C \in \mathbb{R}^{2m \times 2k}$ , and  $D \in \mathbb{R}^{2m \times 2m}$ . We now consider a MIMO system where leaders act as a controller for the followers of the system and are themselves controlled by an external control input  $v = (v^1, \dots, v^m) = f(x_f, \dot{x}_f, x_l, \dot{x}_l)$ . The flow chart of this system is shown in Fig. 1.

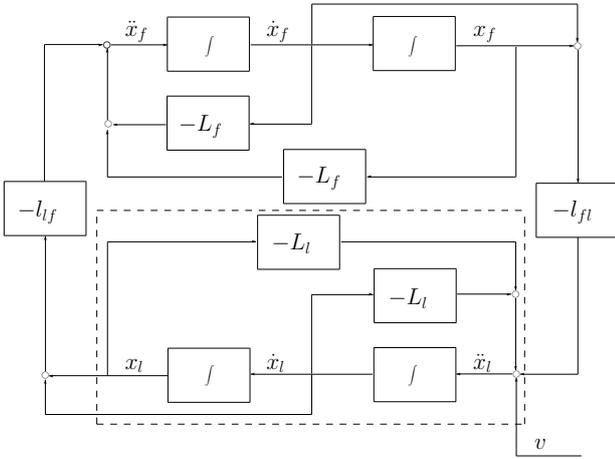


Fig. 1. Flow chart of the double integrator leader-follower control system.

One of the main challenges of the leader-follower setup is to determine agents that can act as leaders. Clearly, only such agents qualify as leaders that can make the system controllable. For agents with single integrator dynamics it is generally assumed (Tanner (2004), Björkenstam et al. (2006)) that  $\dot{x}_l = v$  can be taken to be a free control input, which leads to a problem that only depends on the controllability of  $(A, B)$  (Tanner (2004)). This is no longer possible with double integrator dynamics, since we can no longer assume and we must instead study if the whole network with the given dynamics is controllable through  $v$ . The dependence between the leader and the follower dynamics is illustrated in Fig. 1. One of our main results is that with the current choice of  $F$   $v$  is, indeed, sufficient in order to control the system under the condition that  $(A, B)$  controllable. This result is presented in Section 4.1. In Section 4.2 we further derive topological conditions for the controllability of the follower system, i. e. of  $(A, B)$  and relate them to the single integrator case.

#### 4. MAIN RESULT

We will now show under which conditions the leader-follower network of agents that run a consensus protocol can be controlled through the leaders.

##### 4.1 Controllability Via External Input

Denote the system matrix by

$$\Delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (10)$$

and the input matrix by

$$\Gamma = \begin{pmatrix} 0 \\ F \end{pmatrix}. \quad (11)$$

where the matrices are given by (6)-(9). Note that with the current choice of  $F$  the system  $(D, F)$  is always controllable. It is particularly independent of the form of  $L_l$ .

We obtain the following result for the controllability of the leader-follower system  $(\Delta, \Gamma)$ . Note that we define controllability in the usual sense, i. e. we say that the system is controllable if for any initial state there exists a (free) control input that transfers the initial state to any final state in finite time.

*Lemma 1.* System  $(\Delta, \Gamma)$  is controllable if and only if  $(A, B)$  is controllable.

*Proof.* In order to prove Lemma 1 we will use the Hautus test. Remember that system  $(\Delta, \Gamma)$  is controllable through  $v$  if and only if

$$\text{rank} \mathcal{C}(s) = 2n \quad \forall s \in \mathbb{C}, \quad (12)$$

where

$$\mathcal{C}(s) = \begin{pmatrix} sI - A & -B & 0 \\ -C & sI - D & F \end{pmatrix} \quad (13)$$

i. e. if the rows of (13) are linearly independent. We see directly that if  $(A, B)$  is not controllable, then (13) will be rank deficient for some  $s$ . Hence follows the necessity. For sufficiency we must study the structure of (13). With the matrices (6)-(9) we obtain

$$\mathcal{C}(s) = \left( \begin{array}{cc|cc|c} sI & -I & 0 & 0 & 0 \\ L_f & L_f + sI & l_{lf} & l_{lf} & 0 \\ 0 & 0 & sI & -I & 0 \\ l_{fl} & l_{fl} & L_l & L_l + sI & I \end{array} \right). \quad (14)$$

We see directly that the rows in the last block row are linearly independent of all the other rows. It thus remains to show that for any  $s \in \mathbb{C}$  and any vector  $w = (w_1, w_2, w_3) \in \mathbb{C}^{1 \times 2n}$

$$w \begin{pmatrix} sI & -I & 0 & 0 \\ L_f & L_f + sI & l_{lf} & l_{lf} \\ 0 & 0 & sI & -I \end{pmatrix} = 0 \quad (15)$$

implies  $w = 0$ . Writing (15) as a set of linear equations we obtain

$$w_1 s + w_2 L_f = 0 \quad (16)$$

$$-w_1 + w_2 (L_f + sI) = 0 \quad (17)$$

$$w_2 l_{lf} + w_3 s = 0 \quad (18)$$

$$w_2 l_{lf} - w_3 = 0 \quad (19)$$

For all  $s \neq -1$ , (18) and (19) imply  $w_3 = 0$ . Clearly,  $(w_1, w_2, 0)\mathcal{C}(s) = 0$  if and only if

$$(w_1 \ w_2) (sI - A|B) = 0. \quad (20)$$

The latter implies  $(w_1, w_2)$  as  $(A, B)$  is controllable.

For  $s = -1$ , (16) and (17) become

$$-w_1 + w_2 L_f = 0 \quad (21)$$

$$-w_1 + w_2 L_f - w_2 = 0. \quad (22)$$

Therefore  $w_2 = 0$ ,  $w_1 = 0$ , and, because of (19),  $w_3 = 0$ .  $\square$

#### 4.2 Controllability of the Follower System

Having shown that the leader-follower system with double integrator dynamics is controllable if and only if  $(A, B)$  is controllable, we now derive controllability conditions for the follower system. Recall that  $(A, B)$  is controllable if and only if  $\text{rank}(\mathcal{D}) = 2k$ , where

$$\mathcal{D} = [B \ AB \ A^2 B \ \dots \ A^{2k-1} B] \quad (23)$$

is the controllability matrix. Obviously

$$\left( A, B = \begin{pmatrix} 0 & 0 \\ -l_{lf} & -l_{lf} \end{pmatrix} \right) \quad (24)$$

is controllable if and only if

$$\left( A, b = \begin{pmatrix} 0 \\ -l_{lf} \end{pmatrix} \right) \quad (25)$$

is controllable.

Let us first recall the necessary and sufficient conditions for controllability of  $(L_f, l_{lf})$  found by Tanner (2004):

*Lemma 2.* (Tanner (2004)). System  $(L_f, l_{lf})$  is controllable if and only if all eigenvalues of  $L_f$  are distinct and  $l_{lf}$  is not orthogonal to the eigenvectors of  $L_f$ .

These conditions have been formulated for one leader, i. e. for the case when  $l_{lf}$  is a vector. We present the following extension of Lemma 2 to the leader-follower system with  $m$  leaders and  $k$  followers and double integrator dynamics.

*Lemma 3.* Consider a system of  $m$  leader and  $k$  follower agents, where  $m + k = n$  is the number of agents in the network. Let the Laplacian  $L$  of the system be partitioned as in (3) and the matrices  $A, B$  of the multi-agent system be given by (6-7). The system  $(A, B)$  is controllable if and only if no eigenvector of  $L_f$  is simultaneously orthogonal to all columns of  $l_{lf}$ , and  $L_f$  has at most  $p$  identical eigenvalues where  $p \leq m$  is the number of linearly independent columns of  $l_{lf}$ . Particularly, the system  $(A, B)$  is controllable with one leader if and only if  $l_f$  is not orthogonal to any eigenvector of  $L_f$  and all eigenvalues of  $L_f$  are distinct.

*Proof.* To prove Lemma 3 we will again use the Hautus test. We first show sufficiency of the conditions in Lemma 3.

We know that  $(A, B)$  is controllable if and only if  $(A, b)$  is controllable, where  $b = (0^{1 \times k}, -l_{lf}^T)^T$ . We obtain the following rank condition:

$$\text{rank} \tilde{\mathcal{C}}(s) = 2k \quad \forall s \in \mathbb{C} \quad (26)$$

where

$$\tilde{\mathcal{C}}(s) = (sI - A | b) \quad (27)$$

if and only if  $(A, b)$  controllable. Inserting  $A$  and  $b$ , we obtain

$$\tilde{\mathcal{C}}(s) = \begin{pmatrix} sI & -I & 0 \\ L_f & sI + L_f & -l_{lf} \end{pmatrix}. \quad (28)$$

Since  $-L_f$  is a symmetric matrix, we can make use of the transformation to diagonal form,  $-L_f = UVU^T$ , where  $U$  is the orthonormal matrix of all the eigenvectors of  $-L_f$  and  $V = \text{diag}(-\lambda_i)$ , the diagonal matrix of the real eigenvalues of  $L_f$ . Then we obtain

$$\tilde{\mathcal{C}}(s) = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \underbrace{\begin{pmatrix} sU^T & -U^T & 0 \\ VU^T & (sI + V)U^T & U^T l_{lf} \end{pmatrix}}_{\tilde{\mathcal{C}}} \quad (29)$$

Clearly,  $\tilde{\mathcal{C}}(s)$  has full rank if and only if  $\tilde{\mathcal{C}}(s)$  has full rank. Consider a vector  $w = (w_1, w_2)$ ,  $w \in \mathbb{C}^{1 \times 2k}$ .  $\tilde{\mathcal{C}}(s)$  has full rank if and only if for any  $s \in \mathbb{C}$

$$w \tilde{\mathcal{C}}(s) = 0 \quad (30)$$

implies that  $w = 0$ . Writing (30) as a set of linear equations we obtain

$$(w_1 s + w_2 V) U^T = 0 \quad (31)$$

$$(-w_1 + w_2 (sI + V)) U^T = 0 \quad (32)$$

$$w_2 U^T l_{lf} = 0 \quad (33)$$

and  $(A, b)$  is controllable if and only if (31-33) has no nontrivial solution. We will now show that the conditions postulated in the Lemma are both sufficient and necessary for (31-33) to have  $w = 0$  as the only solution.

(*sufficiency*) Since  $U^T$  is an orthonormal matrix it is invertible. Therefore we can equivalently write (31) and (32) as

$$w_1 s + w_2 V = 0 \quad (34)$$

$$-w_1 + w_2 (sI + V) = 0. \quad (35)$$

Combining (34) and (35) we obtain

$$w_2 \underbrace{(V + Vs + Is^2)}_{Z(s)} = 0 \quad (36)$$

Therefore it must hold that either  $w_2 = 0$ ,  $Z(s) = 0$  or  $w_2 \perp Z(s)$ , where

$$Z(s) = \text{diag}(v_i + v_i s + s^2 = 0). \quad (37)$$

We see directly that  $Z(s) \neq 0 \forall s \in \mathbb{C}$  if at least two entries of  $V$  are distinct. If  $w_2 = 0$  then with (34) we obtain  $w = 0$ , which is the trivial solution. Consider now the condition that  $w_2 \perp Z(s)$ . Since  $Z(s)$  is diagonal, the orthogonality condition can be satisfied if and only if at least one entry of  $Z(s)$  is zero.

Assume that all entries of  $V$  are distinct. Then for any  $i \in \{1, \dots, k\}$  there is an  $s_i$  such that  $v_i + v_i s_i + s_i^2 = 0$ , i. e. the  $i$ th diagonal entry of  $Z(s_i)$  is zero. On the other hand we see directly that the other diagonal entries of  $Z(s_i)$  are nonzero. I. e.  $w_2 Z(s_i) = 0$  if and only if  $w_2 = (0, \dots, 0, \alpha_i, 0, \dots, 0)$  with  $\alpha_i \in \mathbb{C}$  in the  $i$ th place. With this (33) becomes

$$(0 \dots 0 \ \alpha_i \ 0 \dots 0) U^T l_{lf} = 0 \quad (38)$$

which is satisfied if and only if the  $i$ th row of  $U^T l_{lf}$  is zero. Therefore if all entries of  $V$  are distinct,  $(A, b)$  is controllable if and only if no eigenvector of  $L_f$  is simultaneously orthogonal to all columns of  $l_{lf}$ .

(*necessity*) We will now show that if  $(A, b)$  is controllable and  $\text{rank}(l_{lf}) = p$ , then  $V$  must not have more than  $p$  identical eigenvalues. This will be done by contradiction.

If  $(A, b)$  is controllable, then it holds that the only solution to (31-33) is  $w = 0$ . First let two entries of  $V$  be the same. WOLG let  $v_i = v_j$  for some  $i \neq j$ . Then we see directly that there is an  $s_j$  such that two rows of  $Z(s_j)$  are identically zero and we obtain  $w_2 = (0, \dots, 0, \alpha_i, 0, \dots, 0, \alpha_j, 0, \dots, 0)$  with  $\alpha_i, \alpha_j \in \mathbb{C}$ . With this (33) becomes

$$(0 \dots 0 \ \alpha_i \ 0 \dots 0 \ \alpha_j \ 0 \dots 0) U^T l_{lf} = 0. \quad (39)$$

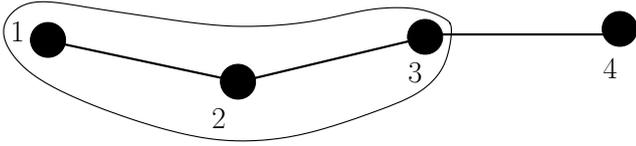


Fig. 2. A path graph. The formation is controllable through agent 4.

If  $p = 1$ , i. e.  $l_{lf}$  has only one linearly independent column, then we can always find some  $\alpha_i, \alpha_j$  such that (39) is satisfied, which is a contradiction to the assumption that  $w = 0$  is the only solution. On the other hand, if  $l_{lf}$  has  $1 < p \leq k$  linearly independent columns, then  $\text{rank}(U^T l_{lf}) = p$  and (39) does not have a nontrivial solution.

Now let  $p \geq 2$ , i. e.  $l_{lf}$  has  $p$  linearly independent columns. Since  $U$  has full rank,  $\text{rank}(U^T l_{lf}) = \text{rank}(l_{lf}) = p$ ,  $1 \leq p \leq k$ . Rewriting (33) as

$$w_2 \perp U^T l_{lf}, \quad (40)$$

we see that (40) can be satisfied by a  $w_2 \neq 0$  if and only if  $w_2$  has at least  $p + 1$  degrees of freedom. As  $w_2$  must also satisfy (36) for some  $s_j \in \mathbb{C}$ , this means that  $Z(s)$  has at least  $p + 1$  zero diagonal entries for some  $s_j$ . This is possible only if  $V$  has at least  $p + 1$  identical eigenvalues. Therefore if  $(A, b)$  is controllable and  $l_{lf}$  has  $p$  linearly independent columns, then  $V$  must not have more than  $p$  identical eigenvalues.  $\square$

*Remark 1.* For a slightly different problem formulation, the necessity of Lemma 3 has been stated independently in Jiang et al. (2009) without proof.

We have therefore shown that controllability of the double order leader-follower network depends only on the topology of the communication graph. This particularly means that the topological conditions found in Rahmani et al. (2009), Ji et al. (2009) and Borsche and Attia (2010) for single leader systems can be applied to systems with double integrator dynamics.

*Example 1.* Consider the path graph with four agents shown in Fig. 2. The Laplacian of the graph is given by

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad (41)$$

Choosing agent 4 as the leader and agents 1-3 as the followers we obtain

$$L_f = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad l_{lf} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad (42)$$

The controllability matrix of this system is then given by

$$\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -3 & 6 & -9 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 6 & -9 & 9 \\ 1 & -2 & 3 & -4 & 5 & -5 \end{bmatrix} \quad (43)$$

which has full rank.

This corresponds with the result of Tanner (2004) that a path graph is always controllable in the single integrator case.

*Example 2.* Consider the complete graph with four agents shown in Fig. 3. The Laplacian of this graph is given by

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}. \quad (44)$$

Choosing agent 4 as the leader and agents 1-3 as the followers we obtain

$$L_f = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \quad l_{lf} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}. \quad (45)$$

The eigenvalues of  $L_f$  are given by  $\{1, 4, 4\}$ , i. e.  $L_f$  is not controllable with one leader according to 3. Indeed we obtain

$$\mathcal{D} = \begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{bmatrix} \quad (46)$$

which has rank 2. We further see that choosing two leaders does not make the system controllable, as  $\text{rank} l_{lf} = 1$  for any choice. This corresponds with the result of Tanner (2004) that the complete graph is uncontrollable by less than  $k - 1$  leaders in the single integrator case.

## 5. CONCLUSION

In this paper we have introduced the multi-agent leader-follower system with double integrator dynamics. We have shown that the necessary and sufficient conditions for a group of followers to be controllable by a group of leaders are the same as for agents with single integrator dynamics. This particularly means that all conditions on the communication topology that have been formulated for agents with single integrator dynamics apply to agents with double integrator dynamics as well. We have furthermore shown that the consensus value of the formation can be controlled via the leaders independently of the fact that the movement of the leaders is restricted by their dynamics.

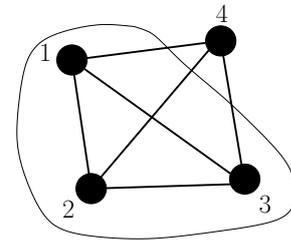


Fig. 3. A complete graph. The formation is uncontrollable with less than three leaders.

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