

A new stability result for switched linear systems

Yashar Kouhi, Naim Bajcinca, Jörg Raisch, and Robert Shorten

Abstract—In this paper we extend the result by Shorten and Narendra on common quadratic Lyapunov functions for pairs of matrices in companion form. Specifically, we show that their result extends to more general matrix pairs provided that an associated transfer function matrix is symmetric. Examples are given to illustrate the usefulness of this result.

I. INTRODUCTION

Consider the switched linear system

$$\dot{x} = A_{i(t)} x, \quad i(t) \in \{1, 2\}, \quad (1)$$

where both A_1, A_2 are constant Hurwitz matrices, that is their eigenvalues lie in the open left half of the complex plane. Let P be a symmetric positive definite matrix such that

$$A_1^T P + P A_1 < 0, \quad (2)$$

$$A_2^T P + P A_2 < 0. \quad (3)$$

Then, the function $V(x)$ is said to be a common quadratic Lyapunov function (CQLF) for the switched system (1). We wish to determine conditions on A_1 and A_2 such that a CQLF exists.

Such stability problems arise in a variety of applications; see for example [6], [7], [8]. Quadratic stability is important for a number of reasons; in particular, quadratically stable systems are robust with respect to discretization, a fact which is important for control design and simulation [11].

An important problem in this context is to specify tractable conditions in terms of A_1 and A_2 to determine whether or not such a P exists. An initial result in this direction was given in [15] where it was shown that any two matrices which are Hurwitz and differ by a rank-1 matrix, will admit a common solution P to the above inequalities provided that the matrix product $A_1 A_2$ has no real negative eigenvalues. Despite much effort it has not been possible to develop similar results for more general matrix pairs [5]. This is due to the fact that boundary of the cone of P matrices associated with a given A matrix plays a crucial role in the solution of a Lyapunov equation [14]. Consequently, the more general common P problem has proved to be

Y. Kouhi, N. Bajcinca, and J. Raisch are with Technische Universität Berlin, Control Systems Group, Einsteinufer 17, Sekr. EN 11, 10587 Berlin, Germany; and with Max Planck Institute for Dynamics of Complex of Technical Systems, Sandtorstr.1, 39106 Magdeburg, Germany (kouhi@control.tu-berlin.de), (bajcinca@mpi-magdeburg.mpg.de) (raisch@control.tu-berlin.de)

R. Shorten is with IBM Research Ireland (robshort@ie.ibm.com).

This work was in part supported by Science Foundation Ireland grant 11/PI/1177, and the FP7 Network of Excellence HYCON2.

considerably more difficult to solve.

An alternative approach therefore is to seek pairs of matrices for which CQLF exist. In this paper we identify one such class; namely pairs of stable matrices from which a symmetric transfer function matrix can be derived.

Our brief paper is structured as follows. The note follows closely [15] in spirit, albeit for a more general system class. First we give preliminary results and definitions. Then we present our main result. Finally, we give examples to illustrate our result.

II. MATHEMATICAL PRELIMINARIES

In this section we present several general results and definitions that are useful in proving the principal result of this note. Throughout this note \mathbb{R} denotes the field of real numbers. We denote n -dimensional real Euclidean space by \mathbb{R}^n and the space of $n \times n$ matrices with real entries by $\mathbb{R}^{n \times n}$. We denote a state space representation of the $m \times m$ transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ by (A, B, C, D) , where we always assume that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ has full column rank, $C \in \mathbb{R}^{m \times n}$ has full row rank, and $D \in \mathbb{R}^{m \times m}$ for some $n \geq m$. Finally, we denote n dimensional identity matrix by I_n .

As discussed in the introduction, we are interested in determining the existence of a common P for inequalities (2) and (3). We now give definitions and basic results which are useful in this context.

- (i) **Strict positive realness:** An $m \times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is said to be strictly positive real (SPR) if there exists an $\alpha > 0$ such that if $G(s)$ is analytic in the region of complex plane which includes all s for which $\text{Re}(s) \geq -\alpha$ and

$$G(j\omega - \alpha) + G^T(-j\omega - \alpha) \geq 0, \quad \forall \omega \in \mathbb{R}; \quad (4)$$

see [13]. The following characterization, inspired principally by Narendra and Taylor [9], provides a more convenient description of a strictly positive real transfer function matrix.

Lemma 1: (See [2]) Given a Hurwitz matrix A . An $m \times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is strictly positive real if and only if

$$G(j\omega) + G^T(-j\omega) > 0, \quad \omega \in \mathbb{R}, \quad (5)$$

and

$$\lim_{\omega \rightarrow \infty} \omega^{2(m-p)} \det\{G(j\omega) + G^T(-j\omega)\} > 0, \quad (6)$$

where $p = \text{rank}\{G(\infty) + G^T(\infty)\}$.

- (ii) **Kalman-Yakubovic-Popov Lemma (KYP):** A basic result in system theory is the KYP lemma. The KYP lemma gives algebraic conditions for the existence of a certain type of Lyapunov functions, see [1], and [16]. In some situations it can be used to determine the existence of a CQLF for a pair of matrices. The classical version of the KYP lemma is now given.

Lemma 2: (Kalman-Yakubovic-Popov Lemma, see [16]) Define $G(s)$ as defined in Lemma 1, and let (A, B) be controllable and (A, C) be observable, then $G(s)$ is SPR if and only if there exist matrices $P = P^T > 0$, Q and W , and a number $\alpha > 0$ satisfying

$$A^T P + PA = -Q^T Q - \alpha P \quad (7)$$

$$B^T P + W^T Q = C \quad (8)$$

$$D + D^T = W^T W. \quad (9)$$

The controllability and observability assumptions can be relaxed in most situations [10], though we do not concern ourselves with such situations here. Furthermore, in some situations the KYP lemma is known to give necessary and sufficient condition for a CQLF for the dynamic systems $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$ where D is invertible and both LTI systems are Hurwitz. A particular case when this is true is when B and C are vectors and D is a scalar, see [15].

- (iii) **Symmetric transfer function matrix:** We now define the class of systems which are of principal interest in this note; namely the class of matrices defined by the following lemma.

Lemma 3: Given a state space representation (A, B, C, D) with a Hurwitz matrix A . Then, the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is symmetric, i.e. $G(s) = G^T(s)$ if and only if $D = D^T$, and

$$CA^i B = (CA^i B)^T, \quad i = 0, 1, \dots, n-1. \quad (10)$$

Proof: By referring to ([3], pp.67), and by defining the characteristic polynomial of A as

$$\det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n,$$

then, the following identity holds

$$C(sI - A)^{-1}B = \frac{1}{\det(sI - A)} [s^{n-1}(CB) + s^{n-2} \times (CAB + a_1 CB) + \dots + (CA^{n-1}B + \dots + a_{n-1}CB)].$$

It is immediately evident that the condition (11) and the assumption $D = D^T$ are sufficient and necessary to show that the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is symmetric. \square

We say that the dynamic system $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$ where the matrices A , B , C , and D satisfy the property of Lemma 3 are **generators of the symmetric transfer function matrix**.

III. MAIN RESULTS

We now present the main result of this note.

Theorem 1: Given two Hurwitz matrices A and $A - BD^{-1}C$ with (A, B) controllable and (A, C) observable, satisfying $D = D^T > 0$ and

$$CA^i B = (CA^i B)^T, \quad i = 0, 1, \dots, n-1; \quad (11)$$

namely both LTI systems associated with these system matrices are generators of a symmetric transfer function matrix. Then, the switched system

$$\dot{x} = (A - \sigma(t)BD^{-1}C)x, \quad \sigma(t) \in \{0, 1\}, \quad (12)$$

is quadratically stable if and only if $A(A - BD^{-1}C)$ has no real negative eigenvalue.

Proof: The proof consists of two parts. First, we discuss the necessity part of the proof. Afterwards, we give the sufficiency part.

Part A (Necessity): Suppose that $\dot{x} = Ax$, $\dot{x} = (A - BD^{-1}C)x$ share a common quadratic Lyapunov function (CQLF). Then, by pre-multiplying the inequality $A^T P + PA < 0$ by the non-singular matrix A^{-T} , and post-multiplying it by A^{-1} , we get

$$A^{-T} P + PA^{-1} < 0. \quad (13)$$

This means that $\dot{x} = A^{-1}x$, $\dot{x} = (A - BD^{-1}C)x$, and, consequently the family of systems

$$\dot{x} = (\omega^2 A^{-1} + (A - BD^{-1}C))x \quad (14)$$

share the same CQLF $V(x) = x^T P x$ for all $\omega \in \mathbb{R}$;

$$[\omega^2 A^{-1} + (A - BD^{-1}C)]^T P + P [\omega^2 A^{-1} + (A - BD^{-1}C)] < 0; \quad (15)$$

as in [15]. Hence, it follows from Lyapunov's second theorem that the matrix $\omega^2 A^{-1} + (A - BD^{-1}C)$ is Hurwitz for all $\omega \in \mathbb{R}$, and thus is non-singular. That is

$$\begin{aligned} \det(\omega^2 A^{-1} + (A - BD^{-1}C)) &\neq 0 \\ \Rightarrow \det(A^{-1}) \det(\omega^2 I + A(A - BD^{-1}C)) &\neq 0. \end{aligned}$$

As A^{-1} is Hurwitz the latter implies that

$$\det(\omega^2 I + A(A - BD^{-1}C)) \neq 0, \quad (16)$$

or equivalently that $A(A - BD^{-1}C)$ has no real negative eigenvalue. Notice that for the proof of necessity the

symmetry conditions given on (11) are not demanded.

Part B (Sufficiency): For sufficiency recall that

$$\frac{1}{2}\{G(\infty) + G^T(\infty)\} = D > 0. \quad (17)$$

Suppose that we are able to show that $G(s)$ is SPR by checking (5), *i.e.*

$$\frac{1}{2}\{G(j\omega) + G^T(-j\omega)\} > 0, \quad (18)$$

then referring to the Kalman-Yakubovic-Popov Lemma, there must exist a matrix $P = P^T > 0$, a scalar $\alpha > 0$ and matrices Q and W satisfying the equations (7-9), respectively. Moreover, the function $V(x) = x^T P x$ is a Lyapunov function for both $\dot{x} = Ax$ and $\dot{x} = (A - BD^{-1}C)x$. In fact, $A^T P + PA < 0$ immediately follows from the equation (7). In the following we show that also $(A - BD^{-1}C)^T P + P(A - BD^{-1}C) < 0$ holds. To see that, we have

$$\begin{aligned} & (A - BD^{-1}C)^T P + P(A - BD^{-1}C) = \\ & = -\alpha P - Q^T Q - C^T D^{-1}(C - W^T Q) \\ & \quad - (C - W^T Q)^T D^{-1} C = \\ & = -\alpha P - (Q - WD^{-1}C)^T (Q - WD^{-1}C) < 0, \quad (19) \end{aligned}$$

where we used the identities (7), (8), and (9). Thus, it remains to demonstrate that (18) holds and $G(s)$ is SPR. To see this, suppose that $A(A - BD^{-1}C)$ has no real negative eigenvalue. By continuity of the $\det(\omega^2 I + A(A - BD^{-1}C))$ with respect to ω everywhere, we can write

$$\begin{aligned} & \det(\omega^2 I + A(A - BD^{-1}C)) > 0 \quad (20) \\ & \Rightarrow \det(\omega^2 I + A^2) \det(I - (\omega^2 I + A^2)^{-1} ABD^{-1}C) > 0 \\ & \Rightarrow \det(\omega^2 I + A^2) \det(I - C(\omega^2 I + A^2)^{-1} ABD^{-1}) > 0 \\ & \Rightarrow \det(\omega^2 I + A^2) \det(D^{-1}) \det(D - C(\omega^2 I + A^2)^{-1} AB) > 0, \end{aligned}$$

where in the third line of the above computation we used the general identity

$$\det(I_n - XY) = \det(I_n - YX).$$

As A has no eigenvalue on $j\omega$ axis, the identity

$$\det(\omega^2 I + A^2) = \det(j\omega I + A) \det(-j\omega I + A),$$

implies that $\det(\omega^2 I + A^2) \neq 0$. Thus, by continuity of $\det(\omega^2 I + A^2)$ with respect to ω everywhere, we can deduce

$$\det(\omega^2 I + A^2) > 0, \quad \forall \omega \in \mathbb{R}.$$

On the other hand, the assumption D positive definite implies that D^{-1} is also positive definite and that $\det(D^{-1}) > 0$. Consequently, by (20) we have that for all $\omega \in \mathbb{R}$

$$\det(D - C(\omega^2 I + A^2)^{-1} AB) > 0. \quad (21)$$

Now, by considering the identity

$$(j\omega I - A)^{-1} A = A(j\omega I - A)^{-1},$$

for the Hurwitz matrix A , the following relationship holds

$$\begin{aligned} & \frac{1}{2} \{ (j\omega I - A)^{-1} + (-j\omega I - A)^{-1} \} = \\ & = \frac{1}{2} (-j\omega I - A)^{-1} \{ (-j\omega I - A) + (j\omega I - A) \} (j\omega I - A)^{-1} \\ & = -(-j\omega I - A)^{-1} A (j\omega I - A)^{-1} \\ & = -(-j\omega I - A)^{-1} (j\omega I - A)^{-1} A \\ & = -(\omega^2 I + A^2)^{-1} A. \end{aligned}$$

Thus, we can write

$$\begin{aligned} & D - C(\omega^2 I + A^2)^{-1} AB = \\ & = D + \frac{1}{2} C \{ (-j\omega I - A)^{-1} + (j\omega I - A)^{-1} \} B = \\ & = \frac{1}{2} \{ D + C(-j\omega I - A)^{-1} B \} + \\ & \quad \frac{1}{2} \{ D + C(j\omega I - A)^{-1} B \} = \\ & = \frac{1}{2} \{ G(j\omega) + G(-j\omega) \}. \quad (22) \end{aligned}$$

Comparing (21) and (22), and using the fact that $G(j\omega)$ is symmetric

$$\det \left(\frac{1}{2} \{ G(j\omega) + G^T(-j\omega) \} \right) > 0. \quad (23)$$

Furthermore, $G(j\omega) + G^T(-j\omega)$ is a Hermitian matrix implying that its eigenvalues are all real. Therefore, if for some frequency it fails to be positive definite then there must exist an $\omega = \omega_1$ such that at least one eigenvalue of the matrix in (18) equals to zero, that is

$$\det \left(\frac{1}{2} \{ G(j\omega_1) + G^T(-j\omega_1) \} \right) = 0.$$

This is obviously in contradiction with (23), and the proof of the Theorem 1 is completed. \square

We now make the following comments, most of which are observed in [15].

Comment 1: Note that since both matrices in the main theorem are assumed to be Hurwitz, it follows that zero can never be in the spectrum of $A(A - BD^{-1}C)$.

Comment 2: It is immediately evident that Theorem 1 also provides a test for strict positive realness. This result is known in Circuit and Systems community as the Hamiltonian half-size test matrix [12]. The implications of this result for switched systems are new.

Comment 3: By congruency the CQLF existence problem for the following pair of Hurwitz LTI systems is equivalent.

$$\begin{aligned} \dot{x} &= A_1 x & ; & \quad \dot{x} = A_2 x \\ \dot{x} &= A_1^{-1} x & ; & \quad \dot{x} = A_2 x \\ \dot{x} &= A_1^{-1} x & ; & \quad \dot{x} = A_2^{-1} x \\ \dot{x} &= A_1 x & ; & \quad \dot{x} = A_2^{-1} x \end{aligned}$$

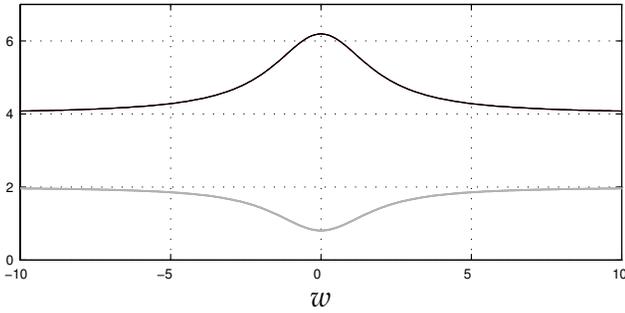


Fig. 1. Eigenvalues of $G(j\omega) + G^T(-j\omega)$ for Example 1

The quadratic stability of all such switched systems is equivalent. Moreover, if a pair is not quadratically stable then a switched system constructed from one of the other pairs can be unstable under certain fast switching policies.

Comment 4: Our main result implies an Aizermann-like [4] statement for the class of the systems considered. Namely, the switched system is stable provided that the pencil $\omega^2 A^{-1} + A - BD^{-1}C$ is Hurwitz for all real ω .

Comment 5: The symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ associated with the matrices $A_1 := A$, and $A_2 := A - BD^{-1}C$ is not unique since different matrices B , C , and D can be chosen for representing the equation $A_2 = A - BD^{-1}C$.

Comment 6: A final contribution of our work is to replace an SPR condition expressed in terms of conditions on transfer function matrices (over an infinite set of frequencies) with a point condition.

IV. EXAMPLES

We now present four examples to illustrate our results.

Example 1: Consider the stable LTI systems

$$\dot{x} = Ax = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x,$$

$$\dot{x} = A_2 x = \begin{bmatrix} -1 & -1 \\ -0.5 & -3 \end{bmatrix} x,$$

where $A - A_2$ has rank 2. Note that with

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

we have that $A_2 = A - BD^{-1}C$. The transfer function matrix

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{(s+2)} \begin{bmatrix} s+1 & 1 \\ 1 & 2(s+3) \end{bmatrix}$$

is symmetric. In addition, it is easily verified that (A, B) and (A, C) are controllable and observable, respectively.

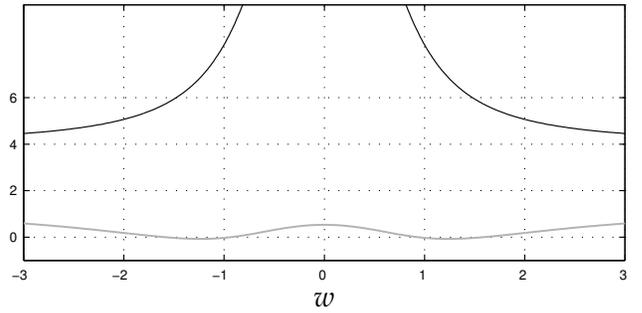


Fig. 2. Eigenvalues of $G(j\omega) + G^T(-j\omega)$ for Example 2.

The eigenvalues of the matrix $G(j\omega) + G^T(-j\omega)$ are depicted in Fig. 1 as functions of ω . It is evident from this figure that $G(s)$ is SPR. Consequently, using the KYP lemma, quadratic stability of the switched system can be deduced.

Alternatively, a much simpler method of establishing the above conclusion is to use the spectral result given in our main theorem. To this end, note that the eigenvalues of the matrix product $A(A - BD^{-1}C)$ are 1.55 and 6.45, respectively. Consequently, from our discussion, $G(s)$ is SPR and the switched system $\dot{x} = (A - \sigma(t)BD^{-1}C)x$ is quadratically stable for $\sigma(t) \in \{0, 1\}$. Using LMI Toolbox, $V(x) = x^T P x$ with P

$$P = \begin{bmatrix} 1.6431 & 0.2821 \\ 0.2821 & 2.6654 \end{bmatrix}, \quad (24)$$

can be computed as an example of such CQLF.

Example 2: Consider now the following two stable LTI systems:

$$\begin{aligned} \dot{x} &= Ax, \\ \dot{x} &= A_2 x, \end{aligned}$$

with

$$A = \begin{bmatrix} -1 & 0 \\ 5 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.7 & -1.38 \\ 8 & -5.2 \end{bmatrix}. \quad (25)$$

$A - A_2$ has rank 2 and with

$$B = \begin{bmatrix} 1 & 2 \\ 10 & 5 \end{bmatrix}, C = \begin{bmatrix} 1 & .2 \\ -1 & .4 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & .625 \end{bmatrix},$$

it follows that $A_2 = A - BD^{-1}C$. Moreover, the transfer function matrix associated to A and A_2 is symmetric

$$G(s) = \frac{1}{(s+1)^2} \begin{bmatrix} 2s^2 + 7s + 6 & 5 + 3s \\ 5 + 3s & .625s^2 + 1.25s + 4.625 \end{bmatrix}.$$

In addition, it is easily verified that (A, B) and (A, C) are controllable and observable.

Fig. 2 depicts part of the eigenvalue loci of $G(j\omega) + G^T(-j\omega)$ as a function of ω . Clearly, the transfer function matrix $G(s)$ is not SPR as the eigenvalue

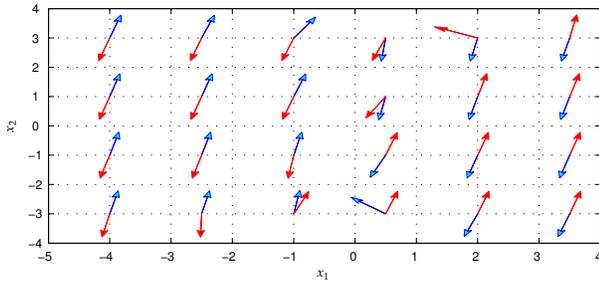


Fig. 3. Direction of the vector fields for the LTI systems $\dot{x} = A_1^{-1}x$ and $\dot{x} = A_2x$ in Example 3

loci is zero at some frequencies. In this case the KYP cannot be used to deduce the existence of a CQLF.

Alternatively, notice the eigenvalues of $A(A - BD^{-1}C)$ are as follows $-2.53, -0.87$. Since the eigenvalues are negative it follows that $G(s)$ cannot be SPR, and also that a CQLF cannot exist. Consequently, the switched system $\dot{x} = (A - \sigma(t)BD^{-1}C)x$ is not quadratically stable for all $\sigma(t) \in \{0, 1\}$.

Example 3: Now, consider the pair of matrices defined in Example 2 and the associated switching system:

$$\dot{x} = A_{i(t)} x(t), \quad A_{i(t)} \in \{A_1, A_2\}. \quad (26)$$

Since the matrix product has a real negative eigenvalue it follows that the determinant of $\omega^2 A_1^{-1} + A_2$ is zero for some finite ω and the vector fields of the systems $\dot{x} = A^{-1}x$ and $\dot{x} = (A - BD^{-1}C)x$ include a point where the angle between the flows is 180° degree, see Fig. 3. Roughly speaking, by switching infinitely fast between the vector fields associated with these matrices in the context of the above switched system, one should arrive at a situation where the state does not converge to the origin for an appropriate initial condition. To verify this we use Floquet theory [4] under the assumption of periodic switching. Note that

$$\begin{aligned} e^{A_1^{-1}t_1} e^{A_2 t_2} &= (I + A_1^{-1}t_1 + \dots)(I + A_2 t_2 + \dots) \\ &\approx I + A_1^{-1}t_1 + A_2 t_2 \end{aligned}$$

has an eigenvalue whose magnitude is greater than unity for small t_1 and t_2 . For example with $t_1 = 0.0016$ and $t_2 = 0.001$ the eigenvalues of the product of exponentials are $1.0001, 0.9932$. Since one of the eigenvalues is greater than unity we have an unstable switching sequence.

Example 4: While it has been shown in the necessity proof of Theorem 1 that having $A_1 A_2$ no real negative eigenvalue is a necessary condition for the existence of a CQLF for Hurwitz matrices A_1 and A_2 , this condition cannot solely guarantee the sufficiency for general pair of Hurwitz matrices. An example that necessary condition fails to be sufficient

is as follows:

$$A_1 = \begin{bmatrix} -1.8 & 0.5 & 0.1 & 0.4 \\ 0.1 & -1.8 & 0.1 & 0.9 \\ 0.1 & 0.7 & 0.6 & -2.1 \\ 0.1 & 0.8 & 0.9 & -1.8 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.8 & 0.1 & 0.9 & 0.3 \\ 0.2 & -0.4 & -1.7 & 0.3 \\ -0.4 & 0.7 & 1.1 & -2 \\ 0.4 & 0.6 & 0.8 & -2.8 \end{bmatrix}.$$

A_1 and A_2 are both Hurwitz, and the eigenvalues of the product $A_1 A_2$ are in $4.696, 0.6303 \pm 1.5505i, .2333$. The spectrum condition is satisfied as the eigenvalues of the product are not real negative. However, by using numerical software it can be checked that no CQLF for this pair exists.

V. CONCLUSIONS

In this paper we have studied the stability properties of a class of switched linear systems whose system matrices satisfy certain symmetry conditions. We have shown that the results by Shorten and Narendra extend to this system class. Future work will consider the case of D (as in [13] singular).

REFERENCES

- [1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15. PA: SIAM, 1994.
- [2] M. Corless and R. Shorten. On the characterization of strict positive realness for general matrix transfer functions. *IEEE Transactions on Automatic Control*, 55(8):1899–1904, 2010.
- [3] T. Kailath. *Linear Systems*. Prentice Hall, Englewood Cliffs, NJ, 1980.
- [4] H. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.
- [5] C. King and R. N. Shorten. Singularity conditions for the non-existence of a common quadratic Lyapunov function for pairs of third order linear time invariant dynamic systems. *Linear Algebra and its Applications*, 413(1):24–35, 2006.
- [6] D. Liberzon. *Switching in systems and control*. Birkhauser, 2003. Volume in series Systems and Control: Foundations and Applications.
- [7] Daniel Liberzon and A. Stephen Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.
- [8] H. Lin and P.J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2):308–322, 2009.
- [9] K. S. Narendra and J.H. Taylor. *Frequency domain criteria for absolute stability*. New York: Academic, 1973.
- [10] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters*, 28:7–10, 1996.
- [11] F. Rossi, P. Colaneri, and R. Shorten. Padé discretization for linear systems with polyhedral Lyapunov functions. *IEEE Transactions on Automatic Control*, 56(11):2717–2722, 2011.
- [12] A. Semlyen and B. Gustavsen. A half-size singularity test matrix for fast and reliable passivity assessment of rational models. *IEEE Transactions on Power Delivery*, 24(1):345–351, 2009.
- [13] R. Shorten, M. Corless, K. Wulff, S. Klinge, and R. Middleton. Quadratic stability and singular SISO switching systems. *IEEE Transactions on Automatic Control*, 54(11):2714–2718, 2009.
- [14] R. N. Shorten, Oliver Mason, Fiachre O’Cairbre, and Paul Curran. A unifying framework for the SISO circle criterion and other quadratic stability criteria. *International Journal of Control*, 77(1):1–8, 2004.
- [15] R. N. Shorten and K. S. Narendra. On common quadratic Lyapunov functions for pairs of stable LTI systems whose system matrices are in companion form. *IEEE Transactions on Automatic Control*, 48(4):618–621, 2003.
- [16] K. Zhou, J. C. Doyle, and K. Glover. *Robust and optimal control*. Prentice Hall, Upper Saddle River, 1996.