

Constructing (Bi)Similar Finite State Abstractions using Asynchronous l -Complete Approximations

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Abstract—This paper constructs a finite state abstraction of a possibly continuous-time and infinite state model in two steps. First, a finite external signal space is added, generating a so called ϕ -dynamical system. Secondly, the strongest asynchronous l -complete approximation of the external dynamics is constructed. As our main results, we show that (i) the abstraction simulates the original system, and (ii) bisimilarity between the original system and its abstraction holds, if and only if the original system is l -complete and its state space satisfies an additional property.

I. INTRODUCTION

Real life control problems for large scale systems are often very challenging due to numerous interactions between different components and usually tight performance requirements. One way to reduce the complexity of such problems is to introduce different control layers using well defined abstractions. Usually, the top control layer will enforce high level specifications, such as interconnection or safety requirements, typically expressible by regular languages. With this specification type, supervisory control theory (SCT) [11] can be used to synthesize a correct by design controller if the abstracted model can be represented by a regular language, i.e., if it can be realized by a finite state machine.

Motivated by this, Tabuada and Pappas [8], [9], [7] developed finite state abstraction methods generating a regular language representation of the plant model. With the same motivation but independently from their work, the notion of a strongest l -complete approximation was introduced by Moor and Raisch [2], [3], recently generalized by Schmuck and Raisch [5] to strongest *asynchronous* l -complete approximations.

While the existence of simulation or bisimulation relations between l -complete approximations and the original system has not yet been formally investigated, the abstraction techniques by Tabuada and Pappas ensure the existence of such relations between the original and the abstracted plant model. However, in their work, the original system is rewritten into a transition system, previous to the abstraction step. The simulation or bisimulation relation is then ensured to hold between the transition system (not the original model) and its finite state abstraction. Their rewriting step is necessary since simulation relations are naturally defined between models evolving on the same time axis. To overcome this limitation, Schmuck and Raisch [6] introduced ϕ -dynamical systems, a

system model with distinct external and internal signals possibly evolving on different time axes. In [6], different notions of simulation and bisimulation were derived, ensuring that they are, respectively, preorders and equivalence relations for this system class. These relations extend existing notions relating systems with different time scales [1], by ensuring behavioral inclusion or equivalence of external trajectories. ϕ -dynamical systems are able to model abstraction processes or signal aggregation by combining both the original (possibly continuous-time) state dynamics and the corresponding external discrete-time behavior. This can naturally be combined with asynchronous l -complete approximations of the external behavior, generating a finite state abstraction if the external signal space is finite. Therefore, in contrast to the work by Tabuada and Pappas, no intermediate transition system has to be introduced to reason about similarity.

After introducing required notation in Section II, we review the notion of ϕ -dynamical systems in Section III and apply the construction of a strongest asynchronous l -complete approximation to this system class in Section IV. In Section V we briefly review the simulation relations defined in [6] for ϕ -dynamical systems. As our main result, we prove the existence of different simulation relations between the original system and its approximation and derive necessary and sufficient conditions for bisimilarity in Section VI.

Our construction extends the work by Tabuada and Pappas in three ways: (i) simulation relations are established between the original state space dynamics and the abstraction, (ii) the accuracy of the abstracted system can be adjusted during construction without refining the external signal space and (iii) the behavioral framework (e.g., [10]) is used, allowing for infinite trajectories with eventuality properties.

II. PRELIMINARIES

In the behavioral framework (e.g., [10]), a *dynamical system* is given by $\Sigma = (T, W, \mathcal{B})$, consisting of the right-unbounded time axis $T \subseteq \mathbb{R}$, the signal space W and the behavior of the system, $\mathcal{B} \subseteq W^T$, where $W^T := \{w \mid w : T \rightarrow W\}$ is the set of all *signals* evolving on T and taking values in W . Slightly abusing notation, we also write $v \in W^T$ if $v : T \rightarrow W$ is a *partial function*. This is understood to be shorthand for $v \in W^{\text{dom}(v)}$, where $\text{dom}(v) = \{t \in T \mid v(t) \text{ is defined}\}$ is the *domain* of v . Furthermore, $\mathbf{i} : T \rightarrow T$ is the *identity map* s.t. ¹ $\forall t \in T . \mathbf{i}(t) = t$.

Let \mathcal{I} be a bounded interval on T , then $W^{\mathcal{I}} := \{w \mid$

¹Throughout this paper we use the notation “ $\forall .$ ”, meaning that all statements after the dot hold for all variables in front of the dot. “ $\exists .$ ” is interpreted analogously.

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$w : \mathcal{I} \rightarrow W$ is the set of *signals on \mathcal{I}* taking values in W . Furthermore, $w|_{\mathcal{I}}$ is the *restriction* of the map $w : T \rightarrow W$ to the domain \mathcal{I} . $\mathcal{B}|_{\mathcal{I}} \subseteq W^{\mathcal{I}}$ denotes the restriction of all trajectories in \mathcal{B} to \mathcal{I} . Now let $W = W_1 \times W_2$ be a product space. Then the *projection* of a signal $w \in W^T$ to W_1 is given by $\pi_{W_1}(w) := \{w_1 \in W_1^T \mid \exists w_2 \in W_2^T . w = (w_1, w_2)\}$ and $\pi_{W_1}(\mathcal{B})$ denotes the projection of all signals in the behavior to W_1 . Given two signals $w_1, w_2 \in W^T$ and two time points $t_1, t_2 \in T$, the *concatenation* $w_3 = w_1 \wedge_{t_2}^{t_1} w_2$ is given by

$$\forall t \in T . w_3(t) = \begin{cases} w_1(t) & , t < t_1 \\ w_2(t - t_1 + t_2) & , t \geq t_1 \end{cases} \quad (1)$$

where we denote $\cdot \wedge_t^t \cdot$ by $\cdot \wedge_t \cdot$.

Following [10], a system $\Sigma = (T, W, \mathcal{B})$ is *complete* if

$$(\forall t_1, t_2 \in T, t_1 \leq t_2 . w|_{[t_1, t_2]} \in \mathcal{B}|_{[t_1, t_2]}) \Leftrightarrow w \in \mathcal{B}, \quad (2)$$

and, following [5, Def.3], we say that $\Sigma = (T, W, \mathcal{B})$ is *asynchronously l-complete* if

$$\left(\begin{array}{l} w|_{[0, l]} \in \mathcal{B}|_{[0, l]} \\ \wedge \forall t \in T . w|_{[t, t+l]} \in \bigcup_{t' \in T} \mathcal{B}|_{[t', t'+l]} \end{array} \right) \Leftrightarrow w \in \mathcal{B}. \quad (3)$$

Now let X be a set. Then, following [5, Def.1], the system $\Sigma_S = (T, W \times X, \mathcal{B}_S)$ is an *asynchronous state space dynamical system* if

$$\begin{array}{l} \forall (w_1, x_1), (w_2, x_2) \in \mathcal{B}_S, t_1, t_2 \in T . \\ \left[\begin{array}{l} \downarrow \\ (x_1(t_1) = x_2(t_2) \Rightarrow (w_1, x_1) \wedge_{t_2}^{t_1} (w_2, x_2) \in \mathcal{B}_S) \end{array} \right] \end{array} \quad (4)$$

and we say that $\Sigma_S = (T, W \times X, \mathcal{B}_S)$ is an asynchronous state space system for $\Sigma = (T, W, \mathcal{B})$ if $\pi_W(\mathcal{B}_S) = \mathcal{B}$.

A *state machine* is a tuple $P = (X, W, \delta, X_0)$, where X is the state space, W is the signal space, $X_0 \subseteq X$ is the set of initial states and $\delta \subseteq X \times W \times X$ is a next state relation. Then

$$\mathcal{B}_f(P) := \left\{ (w, x) \left| \begin{array}{l} x(0) \in X_0 \\ \wedge \forall t \in T . (x(t), w(t), x(t+1)) \in \delta \end{array} \right. \right\}$$

is the *full behavior induced by P* , and we say that $P = (X, W, \delta, X_0)$ realizes $\Sigma_S = (\mathbb{N}_0, W \times X, \mathcal{B}_S)$ if $\mathcal{B}_f(P) = \mathcal{B}_S$. P is a *finite state machine* if $|X| < \infty$ and $|W| < \infty$.

Now let $T = \mathbb{N}_0$. Then, given time instants $t_1, t_2 \in \mathbb{N}_0$, $t_1 \leq t_2$, the string $w \in W^{[t_1, t_2]}$ is of length $|w|_L = t_2 - t_1 + 1$ and for $t_1 < t_2$ we define $w|_{[t_2, t_1]} := \lambda$, where λ denotes the *empty string* with $|\lambda|_L = 0$. Furthermore, the concatenation of the restrictions $w'_1 = w|_{[0, t_1]}$ and $w'_2 = w|_{[0, t_2]}$, with $t_1, t_2 \in \mathbb{N}_0$, $t_1 \leq t_2$, is defined as the standard concatenation of finite strings, i.e., $w'_1 \cdot w'_2 := (w_1 \wedge_0^{t_1+1} w_2)|_{[0, t_1+t_2+1]}$. Furthermore, for a finite string $w = \nu_0 \nu_1 \dots \nu_l$ we denote the restriction of w by $w|_{\langle i, j \rangle} := \nu_i \dots \nu_j$ with $0 \leq i \leq j \leq l$.

III. ϕ -DYNAMICAL SYSTEMS

The common starting point of methods generating finite state abstractions of a possibly continuous-time and infinite state system is the definition of a finite external signal space Γ . This external signal space can be understood as the information content which needs to be preserved or approximated when interconnecting the system to other

components or when controlling it w.r.t. a given specification and is therefore application-specific. While the evolution of the introduced external variable on Γ is in discrete-time \mathbb{N}_0 , the internal dynamics will still evolve on the original, possibly continuous time axis T . To handle such models with distinct internal and external time axes, we use the notion of ϕ -dynamical systems.

Definition 1 ([6], Def.1): Let $\Sigma = (T, W, \mathcal{B})$ be a dynamical system. Then $\Sigma^\phi = (T, \mathbb{N}_0, W, \Gamma, \mathcal{B}, \mathcal{B}_E, \phi)$ is a ϕ -**dynamical system** if

$$\phi : \mathcal{B} \rightarrow 2^{\Gamma^{\mathbb{N}_0} \times \mathcal{T}}$$

where Γ is an external signal space,

$$\mathcal{T} = \left\{ \tau : T \rightarrow \mathbb{N}_0 \mid \begin{array}{l} \tau \text{ is surjective and} \\ \text{monotonically increasing} \end{array} \right\}$$

is a set of time scale transformations and

$$\mathcal{B}_E = \{ \gamma \in \Gamma^{\mathbb{N}_0} \mid \exists w \in \mathcal{B}, \tau \in \mathcal{T} . (\gamma, \tau) \in \phi(w) \} \quad (5)$$

is the external behavior. Furthermore, $\tau^{-1} : \mathbb{N}_0 \rightarrow 2^T$ denotes the inverse time scale transformation², i.e., $\tau^{-1}(k) = \{t \in T \mid \tau(t) = k\}$. \triangleleft

In a ϕ -dynamical system the map ϕ describes how internal signals are discretized (in space and time) to generate the external behavior \mathcal{B}_E . The concept covers both time-triggered and event-triggered discretization. The following example illustrates how event-triggered discretization can be captured in a ϕ -dynamical system.

Example 1: Consider a dynamical system $\Sigma = (T, W, \mathcal{B})$ with $T = \mathbb{R}_0^+$, $W = \mathbb{R} \cap [-10, 10]$, and $w \in \mathcal{B}$ iff w is continuous and $w(0) \in \{-10, 10\}$. Using $\Gamma = \{m_2, m_1, p_1, p_2\}$ and the sets

$$\begin{array}{ll} I_{m_2} = [-10, -4), & I_{m_1} = (-6, 1), \\ I_{p_1} = (-1, 6), & I_{p_2} = (4, 10], \end{array}$$

the external signals are constructed using a set-valued discretization map $\mathfrak{d} : W \rightarrow 2^\Gamma$ s.t.

$$\forall G \in \Gamma, \nu \in W . G \in \mathfrak{d}(\nu) \Leftrightarrow \nu \in I_G.$$

Out of the many different options to construct ϕ from \mathfrak{d} , we discuss the two maps ϕ_a and ϕ_b as depicted in Fig. 1 - 2. The signal map ϕ_a is constructed s.t. for all $\gamma \in \Gamma^{\mathbb{N}_0}$, $\tau_a \in \mathcal{T}$ and $w \in \mathcal{B}$, it holds that $(\gamma, \tau_a) \in \phi_a(w)$ iff

$$\gamma(0) \in \mathfrak{d}(w(0)), \quad \tau_a^{-1}(0) = \{0\}$$

and for all $k \in \mathbb{N}_0, k > 0$ it holds that

$$\begin{array}{l} \tau_a^{-1}(k) = \text{glb} \{ t \geq \tau_a^{-1}(k-1) \mid w(t) \notin \mathfrak{d}^{-1}(\gamma(k-1)) \} \\ \gamma(k) \in \mathfrak{d}(w(\tau_a^{-1}(k))), \end{array} \quad (6)$$

where glb denotes the greatest lower bound and $\forall G \in \Gamma . \mathfrak{d}^{-1}(G) = I_G$. This construction generates a time scale transformation where different points in $\text{dom}(\tau_a)$ are mapped

²If $\forall k \in \mathbb{N}_0 . |\tau^{-1}(k)| = 1$, by slightly abusing notation, we denote the unique element $t_k \in \tau^{-1}(k)$ by $\tau^{-1}(k)$ itself and write $t_k = \tau^{-1}(k)$.

to different points in \mathbb{N}_0 as depicted in Fig. 1 (middle). We therefore call τ_a a *point to point time scale transformation*. ϕ_a triggers an external event when w leaves its current interval, generating the external signal depicted in Fig. 2. It is easy to see that τ_a can be used to define a *set to point time scale transformation*

$$\tau_b^{-1}(k) = [\tau_a^{-1}(k), \tau_a^{-1}(k+1)), \quad (7)$$

depicted in Fig. 1 (bottom). Every point in T is in the domain of τ_b . Combining the construction of τ_b in (7) with the construction of γ in (6) defines a signal map ϕ_b . The resulting ϕ -dynamical systems $\Sigma_i^\phi = (T, \mathbb{N}_0, W, \Gamma, \mathcal{B}, \mathcal{B}_E, \phi_i)$, $i \in \{a, b\}$ then only differ w.r.t. their timescale transformations included in ϕ_a and ϕ_b . \triangleleft

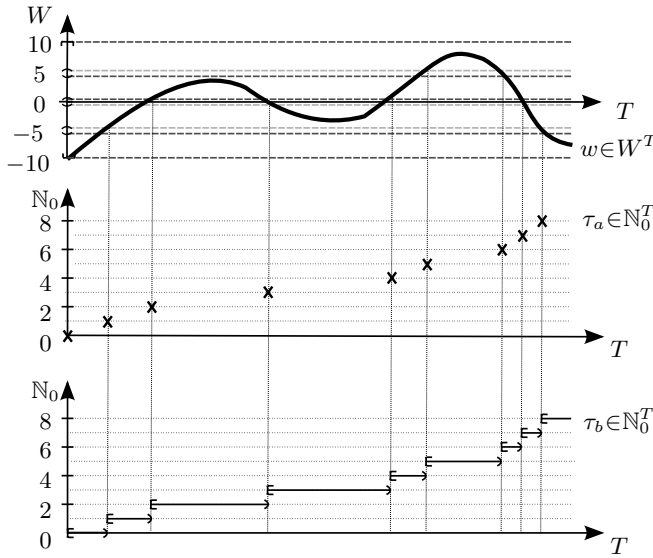


Fig. 1. Illustration of point to point (τ_a) and set to point (τ_b) time scale transformations in Ex.1.

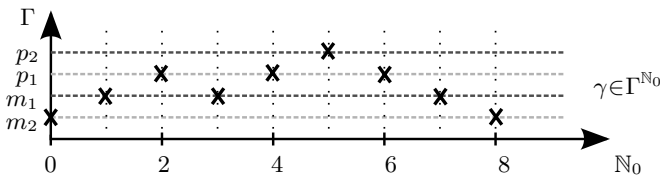


Fig. 2. Illustration of the external signal constructed using event triggered discretization in Ex.1.

States are internal variables for which the axiom of state holds, i.e., all relevant information on the past of the system is captured by those variables. As discussed in [5], two variants of the state property exist for behavioral systems, a synchronous and an asynchronous one. The latter is characterized by (4), the former by (4) if $t = t_1 = t_2$. It will be discussed later in Remark 3 why we restrict attention to asynchronous state space systems. An asynchronous state space ϕ -dynamical system is a system where the asynchronous state property is preserved by the signal map ϕ .

Definition 2 ([6], Def.2): Let $\Sigma^\phi = (T, \mathbb{N}_0, W, \Gamma, \mathcal{B}, \mathcal{B}_E, \phi)$ be a ϕ -dynamical system, X be a set and $\mathcal{B}_S \subseteq (W \times X)^T$.

Then $\Sigma_S^\phi = (T, \mathbb{N}_0, W \times X, \Gamma, \mathcal{B}_S, \mathcal{B}_E, \phi)$ is an **asynchronous state space ϕ -dynamical system** if

$$\forall (w_1, x_1) \in \mathcal{B}_S, (w_2, x_2) \in \mathcal{B}_S, t_1, t_2 \in T, (\gamma_2, \tau_2) \in \phi(w_2), (\gamma_1, \tau_1) \in \phi(w_1), k_1, k_2 \in \mathbb{N}_0 \left. \vphantom{\forall} \right\} \left(\begin{array}{l} x_1(t_1) = x_2(t_2) \\ \wedge k_1 = \tau_1(t_1) \\ \wedge k_2 = \tau_2(t_2) \end{array} \right) \Rightarrow \left(\begin{array}{l} (w_1, x_1) \wedge_{t_2}^{t_1} (w_2, x_2) \in \mathcal{B}_S \\ \wedge (\gamma_1 \wedge_{k_2}^{k_1} \gamma_2, \tau_1 \wedge_{t_2}^{t_1} (\tau_2 + c)) \\ \in \phi(w_1 \wedge_{t_2}^{t_1} w_2) \end{array} \right) \quad (8)$$

where $\forall t \in T$. $c(t) = k_1 - k_2$. Furthermore, Σ_S^ϕ is an asynchronous state space ϕ -dynamical system for Σ^ϕ , if $\pi_W(\mathcal{B}_S) = \mathcal{B}$. \triangleleft

Since possibly not all states are reachable by a state trajectory in $\pi_X(\mathcal{B}_S)$, following [6, Def.5], we define reachable subsets of the state space.

Definition 3: Let $\Sigma_S^\phi = (T, \mathbb{N}_0, W \times X, \Gamma, \mathcal{B}_S, \mathcal{B}_E, \phi)$ be an asynchronous state space ϕ -dynamical system. Then the **internal and external reachable state spaces** $X_I \subseteq X$ and $X_E \subseteq X$, respectively, are defined as

$$X_I := \bigcup_{t \in T} X_I^t \quad \text{and} \quad X_E := \bigcup_{k \in \mathbb{N}_0} X_E^k \quad \text{s.t.} \\ X_I^t := \{ \xi \mid \exists (w, x) \in \mathcal{B}_S . x(t) = \xi \} \quad \text{and} \\ X_E^k := \left\{ \xi \mid \left[\begin{array}{l} \exists (w, x) \in \mathcal{B}_S, (\gamma, \tau) \in \phi(w), t \in \tau^{-1}(k) . \\ \left[\begin{array}{l} x(t) = \xi \end{array} \right] \end{array} \right. \right. \right\}.$$

Now let ζ be a finite string of symbols from Γ . Then $\mathcal{X}_E(\zeta) := \bigcup_{k \geq |\zeta|_L} \mathcal{X}_E^k(\zeta)$ is the set of states compatible with a “recent past” ζ s.t. $\forall k \geq |\zeta|_L$

$$\mathcal{X}_E^k(\zeta) := \left\{ \xi \mid \left[\begin{array}{l} \exists (w, x) \in \mathcal{B}_S, (\gamma, \tau) \in \phi(w), t \in \tau^{-1}(k) . \\ \left[\begin{array}{l} x(t) = \xi \\ \wedge \zeta = \gamma|_{[k-|\zeta|_L, k-1]} \end{array} \right] \end{array} \right. \right\} \quad (9)$$

is the set of states reachable at time t corresponding to external time k and compatible with a “recent past” ζ . \triangleleft

Obviously $\mathcal{X}_E^k(\cdot) \subseteq X_E^k$.

Since the set $\mathcal{X}_E(\cdot)$ will be extensively used in the remainder of this paper, we illustrate its construction by an example.

Example 2: Consider $\Sigma_{S,a}^\phi$ and $\Sigma_{S,b}^\phi$ constructed in Ex.1 and assume $X = W$, i.e., signals $w \in \mathcal{B}$ can be asynchronously concatenated. Then, with $\mathcal{B}_S = \{(x, w) \mid x = w \wedge w \in \mathcal{B}\}$, the systems $\Sigma_{S,i}^\phi = (T, \mathbb{N}_0, W \times X, \Gamma, \mathcal{B}_S, \mathcal{B}_E, \phi_i)$, $i \in \{a, b\}$, are asynchronous state space ϕ -dynamical systems. First let $\zeta = \lambda$, i.e., $|\zeta|_L = 0$, then

$$\mathcal{X}_{E,a}^0(\lambda) = \{-10, 10\} \quad \mathcal{X}_{E,b}^0(\lambda) = [-10, -4] \cup (4, 10]$$

are the sets of states with a “recent past” λ reached at a time t corresponding to external time $k = 0$. Now consider strings $\zeta \in \Gamma = \{m_2, m_1, p_1, p_2\}$, i.e., $|\zeta|_L = 1$, then

$$\mathcal{X}_{E,a}(m_2) = \{-4\} \quad \mathcal{X}_{E,b}(m_2) = (-6, 1) \\ \mathcal{X}_{E,a}(m_1) = \{-6, 1\} \quad \mathcal{X}_{E,b}(m_1) = [-10, -4] \cup (-1, 6)$$

$$\begin{aligned}\mathcal{X}_{E,a}(p_1) &= \{-1, 6\} & \mathcal{X}_{E,b}(p_1) &= (-6, 1) \cup (4, 10] \\ \mathcal{X}_{E,a}(p_2) &= \{4\} & \mathcal{X}_{E,b}(p_2) &= (-1, 4)\end{aligned}$$

are the sets of states compatible with a ‘‘recent past’’ $G \in \Gamma$. Observe that with a point to point time scale transformation only states reached at sampling instances are in $\mathcal{X}_{E,a}(\cdot)$. \triangleleft

IV. l -COMPLETE APPROXIMATIONS

It was shown in [5] that asynchronous l -complete approximations can be used to generate a finite state abstraction of a dynamical system, if it evolves on the discrete time axis \mathbb{N}_0 and the external signal space is finite. If Γ is finite, the external dynamical system $\Sigma_E = (\mathbb{N}_0, \Gamma, \mathcal{B}_E)$, with \mathcal{B}_E as in (5), meets these requirements. Following [2] and [5], a system $\Sigma'_E = (\mathbb{N}_0, \Gamma, \mathcal{B}'_E)$ is an asynchronous l -complete approximation of Σ_E , if (i) Σ'_E is asynchronously l -complete and (ii) $\mathcal{B}'_E \supseteq \mathcal{B}_E$. Σ'_E is a strongest asynchronous l -complete approximation of Σ_E if (i) it is an asynchronous l -complete approximation of Σ_E and (ii) for all other asynchronous l -complete approximations $\Sigma''_E = (\mathbb{N}_0, \Gamma, \mathcal{B}''_E)$ of Σ_E it holds that $\mathcal{B}'_E \subseteq \mathcal{B}''_E$.

Lemma 1 ([5], Lemma 7): Let $\Sigma_E = (\mathbb{N}_0, \Gamma, \mathcal{B}_E)$ be a dynamical system and

$$\mathcal{B}_E^{l\uparrow} := \left\{ \gamma \in \Gamma^{\mathbb{N}_0} \left| \begin{array}{l} \gamma|_{[0,l]} \in \mathcal{B}_E|_{[0,l]} \\ \wedge \forall k \in \mathbb{N}_0 \cdot \gamma|_{[k,k+l]} \in \bigcup_{k' \in \mathbb{N}_0} \mathcal{B}_E|_{[k',k'+l]} \end{array} \right. \right\}.$$

Then $\Sigma_E^{l\uparrow} := (\mathbb{N}_0, \Gamma, \mathcal{B}_E^{l\uparrow})$ is the unique **strongest asynchronous l -complete approximation** of Σ_E . \triangleleft

Since $\Sigma_E^{l\uparrow}$ is asynchronously l -complete, we can use a trivial state space representation saving the last l symbols of the external signal $\gamma \in \mathcal{B}_E$ in the current state.

Lemma 2: Let $\Sigma_E^{l\uparrow} = (\mathbb{N}_0, \Gamma, \mathcal{B}_E^{l\uparrow})$ be an asynchronous l -complete dynamical system,

$$Z := \left(\bigcup_{r \in [0, l-1]} \mathcal{B}_E^{l\uparrow}|_{[0, r-1]} \right) \cup \left(\bigcup_{k \in \mathbb{N}_0} \mathcal{B}_E^{l\uparrow}|_{[k, k+l-1]} \right)$$

be a set and $\mathcal{B}_S^{l\uparrow} \subseteq (\Gamma \times Z)^{\mathbb{N}_0}$ s.t. $(\gamma, z) \in \mathcal{B}_S^{l\uparrow}$ iff

$$z(k) = \begin{cases} \gamma|_{[0, k-1]} & 0 \leq k < l \\ \gamma|_{[k-l, k-1]} & k \geq l \end{cases} \quad \text{and} \quad \gamma \in \mathcal{B}_E^{l\uparrow}.$$

Furthermore, let

$$\phi_l := \mathcal{B}_E^{l\uparrow} \rightarrow 2^{\Gamma^{\mathbb{N}_0} \times \mathcal{T}} \text{ s.t. } \forall \gamma \in \mathcal{B}_E^{l\uparrow} \cdot \phi_l(\gamma) = \{\gamma, \mathbf{i}\}. \quad (10)$$

Then (i) $\Sigma_S^{l\uparrow} := (\mathbb{N}_0, \Gamma \times Z, \mathcal{B}_S^{l\uparrow})$ is an asynchronous state space system for $\Sigma_E^{l\uparrow}$ and (ii) $\Sigma_S^{\phi, l\uparrow} := (\mathbb{N}_0, \mathbb{N}_0, \Gamma \times Z, \Gamma, \mathcal{B}_S^{l\uparrow}, \mathcal{B}_E^{l\uparrow}, \phi_l)$ is an asynchronous ϕ -dynamical state space system for $\Sigma_E^{l\uparrow}$.

Proof: Part (i) is proven by [5, Lemma 3, Lemma 6]. For the second part observe that part (i) implies (4). Now the trivial signal map ϕ_l immediately implies that also (8) holds, since $\mathbf{i} \wedge_{k_2}^{k_1} (\mathbf{i} + (k_1 - k_2)) = \mathbf{i}$. \blacksquare

The state space system $\Sigma_S^{l\uparrow}$ constructed in Lemma 2 is a finite state abstraction of the external behavior of Σ_S^ϕ . Note

that, by construction, $\mathcal{B}_E^{l\uparrow} \supseteq \mathcal{B}_E^{(l+1)\uparrow}$. Hence the parameter l can be used to adjust approximation accuracy.

Observe that $\Sigma_S^{\phi, l\uparrow}$ and $\Sigma_S^{l\uparrow}$ in Lemma 2 exhibit the same external behavior. $\Sigma_S^{\phi, l\uparrow}$ is the trivial transformation of $\Sigma_S^{l\uparrow}$ into the framework of ϕ -dynamical systems. This construction is needed to formally relate the original system Σ_S^ϕ to its finite state abstraction $\Sigma_S^{l\uparrow}$ using the framework of ϕ -dynamical systems as discussed in Sec. V-VI.

As a main result from [5], the abstraction $\Sigma_S^{l\uparrow}$ (and therefore also $\Sigma_S^{\phi, l\uparrow}$) can be realized by a finite state machine (FSM) if $\Gamma < \infty$.

Lemma 3 ([5], Lemma 6): Given the premises of Lemma 2, $|\Gamma| < \infty$, $Z_0 = \{\lambda\}$ and

$$\begin{aligned} \delta = & \left\{ (\zeta, \sigma, \zeta \cdot \sigma) \mid |\zeta|_L < l \wedge \zeta \cdot \sigma \in \mathcal{B}_E^{l\uparrow}|_{[0, |\zeta|_L]} \right\} \\ & \cup \left\{ (\zeta, \sigma, \zeta|_{(1, l-1)} \cdot \sigma) \mid |\zeta|_L = l \wedge \zeta \cdot \sigma \in \bigcup_{k' \in \mathbb{N}_0} \mathcal{B}_E^{l\uparrow}|_{[k', k'+l]} \right\}, \end{aligned}$$

the system $\Sigma_S^{l\uparrow} = (\mathbb{N}_0, \Gamma, Z, \mathcal{B}_S^{l\uparrow})$ is realized by the finite state machine $P = (Z, \Gamma, \delta, Z_0)$. \triangleleft

Note, that for time-variant systems, $\mathcal{B}_E^{l\uparrow}$ and therefore δ may not be computable.

Example 3: Consider the ϕ -dynamical systems $\Sigma_{S,a}^\phi$ and $\Sigma_{S,b}^\phi$ constructed in Ex.1 and let $l = 1$. Then

$$\mathcal{B}_E|_{[0,1]} = \{m_2m_1, p_2p_1\},$$

$$\bigcup_{k' \in \mathbb{N}_0} \mathcal{B}_E|_{[k', k'+1]} = \{m_2m_1, m_1m_2, m_1p_1, p_1m_1, p_1p_2, p_2p_1\}$$

and we can construct $\mathcal{B}_E^{1\uparrow}$ by playing the domino-game depicted in Fig. 3, starting with the domino m_2m_1 or p_2p_1 and always appending a domino from the set $\bigcup_{k' \in \mathbb{N}_0} \mathcal{B}_E|_{[k', k'+1]}$ starting with the last symbol of the previous domino. Observe that for this simple example, $\mathcal{B}_E^{1\uparrow}$ is actually identical to \mathcal{B}_E implying that $\Sigma_E = (\mathbb{N}_0, \Gamma, \mathcal{B}_E)$ is asynchronous 1-complete. Using Lemma 2 we can construct the state space for $\Sigma_S^{1\uparrow}$ and obtain $Z = \{\lambda, m_2, m_1, p_1, p_2\}$. $\Sigma_S^{1\uparrow}$ is realized by the FSM depicted in Fig. 4. \triangleleft

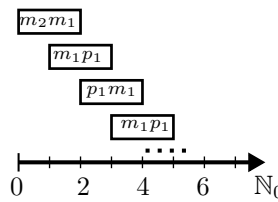


Fig. 3. Domino game generating $\gamma = m_2m_1p_1m_1p_1 \dots$ (Ex.3).

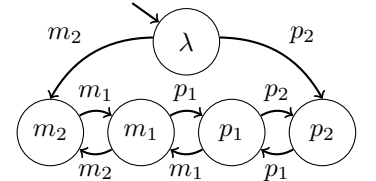


Fig. 4. Finite state machine realizing $\Sigma_S^{1\uparrow}$ (Ex.3).

V. SIMULATION RELATIONS FOR ϕ -DYNAMICAL SYSTEMS

To be able to investigate the existence of simulation relations between the constructed finite state abstraction $\Sigma_S^{\phi, l\uparrow}$ and the original system Σ_S^ϕ , we review the simulation relations for ϕ -dynamical systems introduced in [6].

Definition 4 ([6], Def.4.6): Let $\Sigma_{S,i}^\phi = (T_i, \mathbb{N}_0, W_i \times X_i, \Gamma, \mathcal{B}_{S,i}, \mathcal{B}_{E,i}, \phi_i)$, $i \in 1, 2$ be state space ϕ -dynamical systems. Then a relation $\mathcal{R} \subseteq X_1 \times X_2$ is an **asynchronous simulation relation** from $\Sigma_{S,1}^\phi$ to $\Sigma_{S,2}^\phi$ (written $\mathcal{R} \in \mathfrak{R}_l(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi)$) if

$$\forall \xi_1 \in X_{E,1} \cdot (\exists \xi_2 \in X_{E,2} \cdot (\xi_1, \xi_2) \in \mathcal{R}) \quad (11a)$$

and

$$\forall (w_1, x_1) \in \mathcal{B}_{S,1}, (w', x') \in \mathcal{B}_{S,2}, (\gamma_1, \tau_1) \in \phi_1(w_1), (\gamma', \tau') \in \phi_2(w'), t_1 \in T_1, t_2 \in T_2, k_1, k_2 \in \mathbb{N}_0$$

$$\left(\begin{array}{l} (x_1(t_1), x'(t_2)) \in \mathcal{R} \\ \wedge k_1 = \tau_1(t_1) \\ \wedge k_2 = \tau'(t_2) \end{array} \right) \Rightarrow \left(\begin{array}{l} \exists (w_2, x_2) \in \mathcal{B}_{S,2}, (\gamma_2, \tau_2) \in \phi_2(w_2) \cdot \\ \left(\begin{array}{l} \gamma_2 = \gamma' \wedge_{k_1}^{k_2} \gamma_1 \\ \wedge \forall t \in T_2, t < t_2 \cdot \left(\begin{array}{l} w_2(t) = w'(t) \\ \wedge x_2(t) = x'(t) \\ \wedge \tau_2(t) = \tau'(t) \end{array} \right) \\ \wedge x_2(t_2) = x'(t_2) \\ \wedge \forall k \geq k_2, t'_1 \in \tau_1^{-1}(k - k_2 + k_1), t'_1 > t_1 \cdot \\ \left(\begin{array}{l} \exists t'_2 \in \tau_2^{-1}(k), t'_2 > t_2 \cdot (x_1(t'_1), x_2(t'_2)) \in \mathcal{R} \end{array} \right) \end{array} \right) \end{array} \right) \quad (11b)$$

and an **l -initial simulation relation** from $\Sigma_{S,1}^\phi$ to $\Sigma_{S,2}^\phi$ (written $\mathcal{R} \in \mathfrak{R}_l(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi)$) if

$$\forall \xi_1 \in X_{E,1}^l \cdot (\exists \xi_2 \in X_{E,2}^l \cdot (\xi_1, \xi_2) \in \mathcal{R}) \quad (12a)$$

and (11b) holds. Furthermore, it is an **externally synchronous simulation relation** from $\Sigma_{S,1}^\phi$ to $\Sigma_{S,2}^\phi$ (written $\mathcal{R} \in \mathfrak{R}_l(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi)$) if

$$\forall k \in \mathbb{N}_0, \xi_1 \in X_{E,1}^k \cdot (\exists \xi_2 \in X_{E,2}^k \cdot (\xi_1, \xi_2) \in \mathcal{R}) \quad (13a)$$

and (11b) holds for $k = k_1 = k_2$. Finally, if $T = T_1 = T_2$, then \mathcal{R} is a **synchronous simulation relation** from $\Sigma_{S,1}^\phi$ to $\Sigma_{S,2}^\phi$ (written $\mathcal{R} \in \mathfrak{R}_l(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi)$) if

$$\forall t \in T, \xi_1 \in X_{E,1}^t \cdot (\exists \xi_2 \in X_{E,2}^t \cdot (\xi_1, \xi_2) \in \mathcal{R}) \quad (14a)$$

and (11b) holds for $k = k_1 = k_2$ and $t = t_1 = t_2$. \triangleleft

Analogously to [6, Def.5] we can define four types of simulations and bisimulations.

Definition 5 ([6], Def.5): Let $\Sigma_{S,1}^\phi$ and $\Sigma_{S,2}^\phi$ be state space ϕ -dynamical systems. Then $\Sigma_{S,1}^\phi$ is (i) asynchronously, (ii) externally synchronously, (iii) synchronously, and (iv) l -initially **simulated** by $\Sigma_{S,2}^\phi$, denoted by (i) $\Sigma_{S,1}^\phi \preceq_l \Sigma_{S,2}^\phi$, (ii) $\Sigma_{S,1}^\phi \preceq_{||} \Sigma_{S,2}^\phi$, (iii) $\Sigma_{S,1}^\phi \preceq_{=} \Sigma_{S,2}^\phi$, and (iv) $\Sigma_{S,1}^\phi \preceq_l \Sigma_{S,2}^\phi$, respectively, if there exists an (i) asynchronous, (ii) externally synchronous, (iii) synchronous, and (iv) l -initial simulation relation from $\Sigma_{S,1}^\phi$ to $\Sigma_{S,2}^\phi$, respectively.

Furthermore, $\Sigma_{S,1}^\phi$ and $\Sigma_{S,2}^\phi$ are (i) asynchronously, (ii) externally synchronously, (iii) synchronously, and (iv) l -initially **bisimilar**, denoted by (i) $\Sigma_{S,1}^\phi \cong_l \Sigma_{S,2}^\phi$, (ii) $\Sigma_{S,1}^\phi \cong_{||} \Sigma_{S,2}^\phi$, (iii) $\Sigma_{S,1}^\phi \cong_{=} \Sigma_{S,2}^\phi$, and (iv) $\Sigma_{S,1}^\phi \cong_l \Sigma_{S,2}^\phi$, respectively, if there exists a relation $\mathcal{R} \subseteq X_1 \times X_2$ s.t. \mathcal{R}

and $\mathcal{R}^{-1} = \{(x_2, x_1) \mid (x_1, x_2) \in \mathcal{R}\}$ are (i) asynchronous, (ii) externally synchronous, (iii) synchronous, and (iv) l -initial simulation relations from $\Sigma_{S,1}^\phi$ to $\Sigma_{S,2}^\phi$ and from $\Sigma_{S,2}^\phi$ to $\Sigma_{S,1}^\phi$, respectively. \triangleleft

In contrast to the asynchronous simulation relation, it was shown in [6] that a 0-initial simulation relation is also an (externally) synchronous simulation relation and implies behavioral inclusion.

Lemma 4 ([6], Lemma 1, Thm.1): Let $\Sigma_{S,1}^\phi$ and $\Sigma_{S,2}^\phi$ be state space ϕ -dynamical systems. Then

$$(i) \mathcal{R} \in \mathfrak{R}_0(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi) \Rightarrow \mathcal{R} \in \mathfrak{R}_l(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi)$$

$$(ii) \mathcal{R} \in \mathfrak{R}_0(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi) \Rightarrow \mathcal{R} \in \mathfrak{R}_l(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi)$$

$$(iii) \left(\begin{array}{l} \mathcal{R} \in \mathfrak{R}_0(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi) \\ \wedge T_1 = T_2 = \mathbb{N}_0 \\ \wedge \forall w_1, (\gamma_1, \tau_1) \in \phi_1(w_1) \cdot \tau_1 = \mathbf{i} \\ \wedge \forall w_2, (\gamma_2, \tau_2) \in \phi_2(w_2) \cdot \tau_2 = \mathbf{i} \end{array} \right) \Rightarrow \left(\begin{array}{l} \mathcal{R} \in \mathfrak{R}_l(\Sigma_{S,1}^\phi, \Sigma_{S,2}^\phi) \end{array} \right)$$

$$(iv) (\Sigma_{S,1} \preceq_{l=0} \Sigma_{S,2}) \Rightarrow (\mathcal{B}_{E,1} \subseteq \mathcal{B}_{E,2}) \text{ and}$$

$$(v) (\Sigma_{S,1} \cong_{l=0} \Sigma_{S,2}) \Rightarrow (\mathcal{B}_{E,1} = \mathcal{B}_{E,2}). \quad \triangleleft$$

VI. RELATING THE ORIGINAL SYSTEM AND ITS APPROXIMATION

Now we investigate the existence of simulation and bisimulation relations between an asynchronous state space ϕ -dynamical system and its strongest asynchronous l -complete approximation. Using the results of Lemma 4 we first construct a 0-initial and an l -initial simulation relation from the original system to its abstraction.

Lemma 5: Let $\Sigma_S^\phi = (T, \mathbb{N}_0, W \times X, \Gamma, \mathcal{B}_S, \mathcal{B}_E, \phi)$ be an asynchronous state space ϕ -dynamical system and $\Sigma_S^{\phi, l^\uparrow} = (\mathbb{N}_0, \mathbb{N}_0, \Gamma \times Z, \Gamma, \mathcal{B}_S^{l^\uparrow}, \mathcal{B}_E^{l^\uparrow}, \phi_l)$ an asynchronous ϕ -dynamical state space system for the strongest asynchronous l -complete approximation $\Sigma_E^{l^\uparrow}$ of the discrete external dynamics $\Sigma_E = (\mathbb{N}_0, \Gamma, \mathcal{B}_E)$, constructed in Lemma 2. Furthermore, let

$$\mathcal{R}_0 = \left\{ (\xi, \zeta) \in (X \times Z) \mid \xi \in \left\{ \begin{array}{l} \mathcal{X}_E^{|\zeta|_L}(\zeta) \quad , \quad |\zeta|_L < l \\ \mathcal{X}_E(\zeta) \quad , \quad |\zeta|_L = l \end{array} \right. \right\} \quad (15)$$

$$\mathcal{R}_l = \{ (\xi, \zeta) \in (X \times Z) \mid |\zeta|_L = l \wedge \xi \in \mathcal{X}_E(\zeta) \}. \quad (16)$$

be two relations. Then

$$(i) \mathcal{R}_0 \in \mathfrak{R}_0(\Sigma_S^\phi, \Sigma_S^{\phi, l^\uparrow}) \text{ and}$$

$$(ii) \mathcal{R}_l \in \mathfrak{R}_l(\Sigma_S^\phi, \Sigma_S^{\phi, l^\uparrow}).$$

Proof: See Appendix. \blacksquare

Example 4: Consider the ϕ -dynamical systems $\Sigma_{S,i}^\phi$, $i \in \{a, b\}$ introduced in Ex.1, their reachable state sets $\mathcal{X}_{E,i}(\cdot)$ determined in Ex.2 and the strongest asynchronous 1-complete approximation $\Sigma_{S,1}^{\phi, 1^\uparrow}$ constructed in Ex.3. Using Lemma 5 we know that

$$\mathcal{R}_{1,a} = \{(-4, m_2), (-6, m_1), (1, m_1), (-1, p_1), (6, p_1), (4, p_2)\},$$

$$\mathcal{R}_{1,b} = \{(\xi, m_2) \mid \xi \in (-6, 1)\} \cup \{(\xi, m_1) \mid \xi \in [-10, -4] \cup (-1, 6)\}$$

$$\cup \{(\xi, p_1) \mid \xi \in (-6, 1) \cup (4, 10)\} \cup \{(\xi, p_2) \mid \xi \in (-1, 4)\}$$

are 1-initial simulation relations from $\Sigma_{S,i}^\phi$ to $\Sigma_S^{\phi,1^\uparrow}$. \triangleleft

Since l -complete systems have the same external behavior as their strongest asynchronous l -complete approximations, i.e., $\mathcal{B}_E = \mathcal{B}_E^{l^\uparrow}$, we could guess that the inverse relations of (15) and (16) are 0- and l -initial simulation relations from the abstraction to the original system, if Σ_E is l -complete. However, for \mathcal{R}_0 , observe that the “recent past” of states ξ reached at time $k < l$, i.e. $\xi \in X_E^{k < l}$, has length $k < l$ and is therefore, in general, not sufficient to uniquely determine the future behavior of an l -complete system. Furthermore, even for \mathcal{R}_l , l -completeness of Σ_E is not sufficient, as the following example illustrates.

Example 5: For simplicity consider a discrete system $\Sigma = (\mathbb{N}_0, \Gamma, \mathcal{B}_1)$, $\Gamma = \{a, b, c\}$ realized by the FSM P_1 depicted in Fig. 5 (left). Observe that Σ is asynchronously 1-complete and its strongest asynchronous 1-complete approximation Σ^{1^\uparrow} , realized by the FSM P_2 depicted in Fig. 5 (right), has the same behavior. We can easily generate systems

$$\begin{aligned} \Sigma_{S,1}^\phi &= (\mathbb{N}_0, \mathbb{N}_0, \Gamma \times \{\xi_1, \xi_2, \xi_3\}, \Gamma, \mathcal{B}_f(P_1), \mathcal{B}_1, \phi_l) \\ \Sigma_S^{\phi,1^\uparrow} &= (\mathbb{N}_0, \mathbb{N}_0, \Gamma \times \Gamma, \Gamma, \mathcal{B}_f(P_2), \mathcal{B}_1^{l^\uparrow}, \phi_l) \end{aligned}$$

from Σ and Σ^{1^\uparrow} , respectively, by using the trivial signal map ϕ_l (10). Now using (16) gives the relation $\mathcal{R}_{l=1}^{-1} = \{(a, \xi_1), (a, \xi_3), (b, \xi_2), (c, \xi_2)\}$. It can be easily verified that $\mathcal{R}_{l=1}^{-1}$ is not a 1-initial simulation relation from $\Sigma_S^{\phi,1^\uparrow}$ to $\Sigma_{S,1}^\phi$ since b and c can occur in state a of P_2 while only c can occur in state ξ_3 of P_1 . \triangleleft

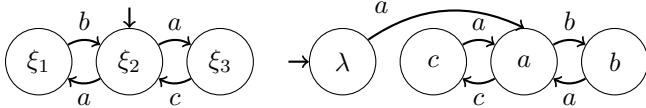


Fig. 5. FSM P_1 (left) and P_2 (right) in Ex.5.

We therefore have to additionally ensure that all states with identical “recent past” allow for the same future external behavior. Inspired by [7, Thm. 4.18] we formulate this property as an l -initial simulation relation \mathcal{R}_X from the original system to itself. The following lemma shows that this condition together with asynchronous l -completeness of the original system (i.e., $\mathcal{B}_E = \mathcal{B}_E^{l^\uparrow}$) is necessary and sufficient for \mathcal{R}_l^{-1} to be a simulation relation from the abstraction to the original system.

Lemma 6: Given the premises of Lemma 5 and

$$\mathcal{R}_X = \{(\xi_a, \xi_b) \in (X \times X) \mid \exists \zeta \in \Gamma^l. \xi_a, \xi_b \in \mathcal{X}_E(\zeta)\} \quad (17)$$

it holds that

$$\left(\begin{array}{l} \mathcal{B}_E = \mathcal{B}_E^{l^\uparrow} \\ \wedge \mathcal{R}_X \in \mathfrak{R}_l(\Sigma_S^\phi, \Sigma_S^\phi) \end{array} \right) \Leftrightarrow \mathcal{R}_l^{-1} \in \mathfrak{R}_l(\Sigma_S^{\phi,1^\uparrow}, \Sigma_S^\phi).$$

Proof: See Appendix. \blacksquare

Example 6: The inverse relations of $\mathcal{R}_{1,a}$ and $\mathcal{R}_{1,b}$ from Ex.4 are a 1-initial simulation relation from $\Sigma_S^{\phi,1^\uparrow}$ to $\Sigma_{S,i}^\phi$, $i \in \{a, b\}$, respectively, since $\Sigma_{S,i}^\phi$ is 1-complete and $\mathcal{R}_X \in \mathfrak{R}_1(\Sigma_{S,i}^\phi, \Sigma_{S,i}^\phi)$ holds. \triangleleft

Remark 1: Observe, that (12a) and the last line of (11b) holds for \mathcal{R}_X by construction, as shown in the proof of Lemma 6. Therefore, requiring $\mathcal{R}_X \in \mathfrak{R}_l(\Sigma_S^\phi, \Sigma_S^\phi)$ only ensures, that all state trajectories starting with the same “recent past” are able to generate the same future external behavior by not changing their “full past”. \triangleleft

Remark 2: Lemma 6 shows that the existence of an l -initial simulation relation from $\Sigma_S^{\phi,1^\uparrow}$ to Σ_S^ϕ implies external behavioral equivalence, i.e., $\mathcal{B}_E = \mathcal{B}_E^{l^\uparrow}$ (extending Lemma 4). \triangleleft

Remark 3: It was shown in [5] that an asynchronous (in contrast to a synchronous) l -complete approximation can be represented by a finite state machine. This is necessary to apply well-known controller synthesis methods (e.g., SCT [11]). Since the focus of this paper is to construct a finite state abstraction for controller synthesis, we have restricted our attention to asynchronous l -complete approximations.

It is easy to show, that the external behavior \mathcal{B}_E of (externally) synchronous state space ϕ -dynamical systems (see [5, Def.2]) is asynchronously l -complete, i.e., $\mathcal{B}_E = \mathcal{B}_E^{l^\uparrow}$, if and only if the system is an asynchronous state space ϕ -dynamical system. Therefore, we are only able to establish the results in Lemma 6 for the latter system class. \triangleleft

As our main result, we now show that a strongest asynchronous l -complete approximation simulates the original system in various ways and that the conditions in Lemma 6 imply bisimilarity of the original system and its approximation.

Theorem 1: Given the premises of Lemma 5 - 6 it holds that **(i)** $\Sigma_S^\phi \preceq_{l=0} \Sigma_S^{\phi,1^\uparrow}$, **(ii)** $\Sigma_S^\phi \preceq_l \Sigma_S^{\phi,1^\uparrow}$, **(iii)** $\Sigma_S^\phi \preceq_{l_1} \Sigma_S^{\phi,1^\uparrow}$, **(iv)** $\left(\begin{array}{l} T = \mathbb{N}_0 \\ \wedge \forall w, (\gamma, \tau) \in \phi(w). \tau = \mathbf{i} \end{array} \right) \Rightarrow \left(\Sigma_S^\phi \preceq_{\mathbf{i}} \Sigma_S^{\phi,1^\uparrow} \right)$, **(v)** $\Sigma_S^\phi \preceq_l \Sigma_S^{\phi,1^\uparrow}$, and **(vi)** $\left(\begin{array}{l} \mathcal{B}_E = \mathcal{B}_E^{l^\uparrow} \\ \wedge \mathcal{R}_X \in \mathfrak{R}_l(\Sigma_S^\phi, \Sigma_S^\phi) \end{array} \right) \Leftrightarrow \left(\begin{array}{l} \mathcal{R}_l \in \mathfrak{R}_l(\Sigma_S^\phi, \Sigma_S^{\phi,1^\uparrow}) \\ \wedge \mathcal{R}_l^{-1} \in \mathfrak{R}_l(\Sigma_S^{\phi,1^\uparrow}, \Sigma_S^\phi) \end{array} \right) \Rightarrow \left(\Sigma_S^\phi \cong_l \Sigma_S^{\phi,1^\uparrow} \right)$.

Proof: **(i)** From Lemma 5 (i) and Def. 5. **(ii)** From (i), Lemma 4 (i) and Def. 5. **(iii)** From (i), Lemma 4 (ii) and Def. 5. **(iv)** From (i), Lemma 4 (iii) and Def. 5. **(v)** From Lemma 5 (ii) and Def. 5. **(vi)** From Lemma 5 (ii), Lemma 6 and Def. 5. \blacksquare

Remark 4: There is a strong connection between Thm. 1 (vi) and the work of Tabuada [7, Thm. 4.18]. It is possible to show that the construction of the quotient system in [7, Def.4.17] coincides with a realization of the strongest asynchronous one-complete approximation if a time-shifted version of the state space construction is used in Def. 2 (similar to [4, Sec. 4]). This time-shift also “shifts” the definition of the sets $\mathcal{X}_E(\cdot)$ and therefore, all relations used in Lemma 5 and 6. Using these time-shifted definitions, it can be shown, that for transition systems the results in [7, Thm. 4.18] and Thm. 1 (vi) are equivalent. \triangleleft

VII. CONCLUSION

We have shown in this paper, that the concepts of asynchronous state space ϕ -dynamical systems and strongest asynchronous l -complete approximations can be combined to generate a finite state abstraction realizable by a finite state machine and therefore suitable for controller synthesis using supervisory control theory. Using simulation relations developed for ϕ -dynamical systems, we have proven that a strongest asynchronous l -complete approximation simulates the original system in various ways. In particular, necessary and sufficient conditions for the existence of a bisimulation relation were derived.

APPENDIX

Proof of Lemma 5 (i):

1) Show (12a) holds for $l = 0$:

- Observe that $Z_E^0 = Z_I^0 = \{\lambda\}$ (from Def. 3 (with state space Z) and the construction of Z in Lemma 2) and $\mathcal{X}_E^0(\lambda) = X_E^0$ (from (9) and Def. 3).
- With the construction of \mathcal{R}_0 in (15), this implies $\forall \xi \in X_E^0 . (\xi, \lambda) \in \mathcal{R}_0$, i.e., (12a) holds for $l = 0$.

2) Show (11b) holds:

- Using the trivial signal map ϕ_l we fix $(w_1, x_1) \in \mathcal{B}_S, (\gamma', z') \in \mathcal{B}_S^{\uparrow}, (\gamma_1, \tau_1) \in \phi(w_1), t_1 \in T, k_1, k_2 \in \mathbb{N}_0$ s.t. the left side of the implication in (12a) holds, i.e., $k_1 = \tau_1(t_1)$ and $(x_1(t_1), z'(k_2)) \in \mathcal{R}_0$.
- Now we construct particular z_2 and γ_2 to show that the right side of the implication in (11b) holds:
►With the construction of \mathcal{R}_0 in (15) we have

$$x_1(t_1) \in \begin{cases} \mathcal{X}_E^{|z'(k_2)|_L}(z'(k_2)) & , |z'(k_2)|_L < l \\ \mathcal{X}_E(z'(k_2)) & , |z'(k_2)|_L = l \end{cases}$$

and therefore we can fix (from (9)) $(\tilde{w}, \tilde{x}) \in \mathcal{B}_S, (\tilde{\gamma}, \tilde{\tau}) \in \phi(\tilde{w}), \tilde{k} \in \mathbb{N}_0, \tilde{t} \in \tau^{-1}(\tilde{k})$ s.t. $x_1(t_1) = \tilde{x}(\tilde{t})$ and $z'(k_2) = \tilde{\gamma}|_{[\max(0, \tilde{k} - |z'(k_2)|_L, \tilde{k} - 1]}$. Furthermore, if $|z'(k_2)|_L < l$ we have $\tilde{k} = |z'(k_2)|_L$.

►Since Σ_S^ϕ is an asynchronous state space dynamical system, $x_1(t_1) = \tilde{x}(\tilde{t})$ implies that we can pick $w'_1 = \tilde{w} \wedge_{t_1}^{\tilde{t}} w_1, x'_1 = \tilde{x} \wedge_{t_1}^{\tilde{t}} x_1, \tau'_1 = \tilde{\tau} \wedge_{t_1}^{\tilde{t}} (\tau_1 + \tilde{k} - k_1)$ and $\gamma'_1 = \tilde{\gamma} \wedge_{k_1}^{\tilde{k}} \gamma_1$ and have $(w'_1, x'_1) \in \mathcal{B}_S$ and $(\gamma'_1, \tau'_1) \in \phi(w'_1)$ implying $\gamma'_1 \in \mathcal{B}_E$.

►Since Σ_E^{\uparrow} is the strongest async. l -complete approx. of Σ_E we have $\mathcal{B}_E \subseteq \mathcal{B}_E^{\uparrow}$ and therefore $\gamma'_1 \in \mathcal{B}_E^{\uparrow}$.

►Now pick $z'_1 \in Z^{\mathbb{N}_0}$ s.t. $\forall k \in \mathbb{N}_0 . z'_1(k) = \gamma'_1|_{[\max(0, k-l, k-1]}$ and observe that the construction of γ'_1 implies $z'_1(\tilde{k}) = z'(k_2)$ and since $\gamma'_1 \in \mathcal{B}_E^{\uparrow}$ it follows from Lemma 2 by construction that $(z'_1, \gamma'_1) \in \mathcal{B}_S^{\uparrow}$.

►Now pick $z_2 := z' \wedge_{\tilde{k}}^{k_2} z'_1$ and $\gamma_2 := \gamma' \wedge_{\tilde{k}}^{k_2} \gamma'_1$. Since $\Sigma_S^{\uparrow} = (\mathbb{N}_0, \Gamma, Z, \mathcal{B}_S^{\uparrow})$ is an asynchronous state space system, $(z_1, \gamma_1), (z'_1, \gamma'_1) \in \mathcal{B}_S^{\uparrow}$ and $z'_1(\tilde{k}) = z'(k_2)$ from above, (4) implies $(z_2, \gamma_2) \in \mathcal{B}_S^{\uparrow}$.

- Finally we show, that for this choice of z_2 and γ_2 the right side of the implication in (11b) holds:

►Show $\gamma_2 = \gamma' \wedge_{k_1}^{k_2} \gamma_1$: Observe that

$$\gamma_2 = \gamma' \wedge_{\tilde{k}}^{k_2} \gamma'_1 = \gamma' \wedge_{\tilde{k}}^{k_2} (\tilde{\gamma} \wedge_{k_1}^{\tilde{k}} \gamma_1) = \gamma' \wedge_{k_1}^{k_2} \gamma_1.$$

►Show $\forall k < k_2 . \gamma_2(k) = \gamma'(k) \wedge z_2(k) = z'(k)$: Follows from (1) and the construction of z_2 and γ_2 .

►Show $z_2(k_2) = z'(k_2)$: From (1) we have $z_2(k_2) = z'_1(\tilde{k})$ and from above $z'_1(\tilde{k}) = z'(k_2)$.

►Show:

$$\forall k \geq k_2, t'_1 \in \tau_1^{-1}(k - k_2 + k_1), t'_1 > t_1 . (x_1(t'_1), z_2(k)) \in \mathcal{R}_0$$

With $\hat{k} := k - k_2 + \tilde{k}$ and $z_2 = z' \wedge_{\tilde{k}}^{k_2} z'_1$ we get

$$\forall \hat{k} \geq \tilde{k}, t'_1 \in \tau_1^{-1}(\hat{k} - \tilde{k} + k_1), t'_1 > t_1 . (x_1(t'_1), z'_1(\hat{k})) \in \mathcal{R}_0.$$

Using $\tau_1'^{-1} = \tilde{\tau}^{-1} \wedge_{k_1}^{\tilde{k}} (\tau_1^{-1} + \tilde{t} - t_1)$, $x'_1 = \tilde{x} \wedge_{t_1}^{\tilde{t}} x_1$ and $\hat{t} := t'_1 + \tilde{t} - t_1$ gives

$$\forall \hat{k} \geq \tilde{k}, \hat{t} \in \tau_1'^{-1}(\hat{k}), \hat{t} > \tilde{t} . (x'_1(\hat{t}), z'_1(\hat{k})) \in \mathcal{R}_0.$$

With the construction of \mathcal{R}_0 and z'_1 the statement to be proven is therefore true iff the statement

$$\forall \hat{k} \geq \tilde{k}, \hat{t} \in \tau_1'^{-1}(\hat{k}), \hat{t} > \tilde{t} . \left\{ \begin{array}{l} \mathcal{X}_E^{\hat{k}}(\gamma'_1|_{[0, \hat{k}-1]}) \quad , \hat{k} < l \\ \mathcal{X}_E(\gamma'_1|_{[\hat{k}-l, \hat{k}-1]}) \quad , \hat{k} \geq l \end{array} \right.$$

holds, what is true from Def. 3, since $(w'_1, x'_1) \in \mathcal{B}_S$ and $(\gamma'_1, \tau'_1) \in \phi(w'_1)$ from above. \blacksquare

Proof of Lemma 5 (ii):

1) Show (12a) holds for l :

- Fix any $(w, x) \in \mathcal{B}_S, (\gamma, \tau) \in \phi(w), t \in \tau^{-1}(l)$.
- Observe that $\gamma \in \mathcal{B}_E$ (from Def. 1), $\gamma \in \mathcal{B}_E^{\uparrow}$ (from Lemma 1) and $x(t) \in X_E^l$ (from Def. 3).
- This implies that we can pick $\zeta = \gamma|_{[0, l-1]}$ and have $\zeta \in Z_E^l$ (from Lemma 2) and $x(t) \in \mathcal{X}_E(\zeta)$ (from (9)), i.e., (12a) holds.

2) Show (11b) holds: This proof is identical to (i.2) only considering states ζ of length l , and is therefore omitted. \blacksquare

Proof of Lemma 6 “ \Rightarrow ”:

1.) Show (12a) holds for \mathcal{R}_l^{-1} :

- Pick $\zeta \in Z_E^l$ and observe, that from Lemma 2 and $\mathcal{B}_E = \mathcal{B}_E^{\uparrow}$ follows that there exists $(w, x) \in \mathcal{B}_S, (\gamma, \tau) \in \phi(w), t \in \tau^{-1}(l)$ s.t. $\gamma|_{[0, l-1]} = \zeta$ implying $x(t) \in X_E^l$ and $x(t) \in \mathcal{X}_E(\zeta)$, i.e., (12a) holds.

2.) Show (11b) holds for \mathcal{R}_l^{-1} :

- Using the trivial signal map ϕ_l we fix $(\gamma_1, z_1) \in \mathcal{B}_S^{\uparrow}, (w', x') \in \mathcal{B}_S, (\gamma', \tau') \in \phi(w'), t_2 \in T, k_1, k_2 \in \mathbb{N}_0$ s.t. the left side of the implication in (11b) holds, i.e., $k_2 = \tau_2(t_2)$ and $(z_1(k_1), x'(t_2)) \in \mathcal{R}_l^{-1}$.
- Now we construct particular x_2, w_2, τ_2 and γ_2 to show that the right side of the implication in (11b) holds:
►Since $(\gamma_1, z_1) \in \mathcal{B}_S^{\uparrow}$ we have $\gamma_1 \in \mathcal{B}_E^{\uparrow}$ from Lemma 2 and as \mathcal{B}_E is l -complete, we have $\mathcal{B}_E = \mathcal{B}_E^{\uparrow}$ and therefore $\gamma_1 \in \mathcal{B}_E$. Using (5), we can therefore fix $(w_1, x_1) \in \mathcal{B}_S, \tau_1 \in T$ s.t. $(\gamma_1, \tau_1) \in \phi(w_1)$.
►Remember $(\gamma_1, z_1) \in \mathcal{B}_S^{\uparrow}$ and $(z_1(k_1), x'(t_2)) \in \mathcal{R}_l^{-1}$. Using (9), we can pick $t_1 \in \tau_1^{-1}(k_1)$ and have $x_1(t_1) \in \mathcal{X}_E(z_1(k_1))$. Using (17) this implies $(x_1(t_1), x'(t_2)) \in \mathcal{R}_X$.

► Since $\mathcal{R}_{\mathcal{X}} \in \mathfrak{R}_l(\Sigma_S^\phi, \Sigma_S^\phi)$ we know that we can pick $(w_2, x_2) \in \mathcal{B}_S, (\gamma_2, \tau_2) \in \phi(w_2)$ s.t.

$$\left(\begin{array}{l} \gamma_2 = \gamma' \wedge_{k_1}^{k_2} \gamma_1 \\ \wedge \forall t \in T_2, t < t_2 . \left(\begin{array}{l} w_2(t) = w'(t) \\ \wedge x_2(t) = x'(t) \\ \wedge \tau_2(t) = \tau'(t) \end{array} \right) \\ \wedge x_2(t_2) = x'(t_2) \\ \wedge \forall k \geq k_2, t'_1 \in \tau_1^{-1}(k - k_2 + k_1), t'_1 > t_1 . \\ \quad \left[\begin{array}{l} \exists t'_2 \in \tau_2^{-1}(k), t'_2 > t_2 . (x_1(t'_1), x_2(t'_2)) \in \mathcal{R}_{\mathcal{X}} \end{array} \right] \end{array} \right) \quad (18)$$

• Observe, that with this choice of $w_2, x_2, \gamma_2, \tau_2$ the first tree lines of the right side of (11b) for \mathcal{R}_l^{-1} are equivalent to the first tree lines in (18).

• Now we show that the last line of (11b) holds for \mathcal{R}_l^{-1} .
► Observe $x_2(t_2) = x'(t_2) \in \mathcal{X}_E(z(k_1))$. Using (9) we can therefore fix $(\tilde{w}, \tilde{x}) \in \mathcal{B}_S, (\tilde{\gamma}, \tilde{\tau}) \in \phi(\tilde{w}), \tilde{k} \in \mathbb{N}_0, \tilde{t} \in \tau^{-1}(\tilde{k})$ s.t. $x_2(t_2) = \tilde{x}(\tilde{t})$ and $z(k_1) = \tilde{\gamma}|_{[\tilde{k}-l, \tilde{k}-1]}$, implying $\tilde{\gamma}|_{[\tilde{k}-l, \tilde{k}-1]} = \gamma_1|_{[k_1-l, k_1-1]}$.

► Since Σ_S^ϕ is an asynchronous state space dynamical system, we can pick $w'' = \tilde{w} \wedge_{t_2}^{\tilde{t}} w_2, x'' = \tilde{x} \wedge_{t_2}^{\tilde{t}} x_2, \tau'' = \tilde{\tau} \wedge_{t_2}^{\tilde{t}} (\tau_2 + \tilde{k} - k_2)$ and $\gamma'' = \tilde{\gamma} \wedge_{k_2}^{\tilde{k}} \gamma_2$ and have $(w'', x'') \in \mathcal{B}_S$ and $(\gamma'', \tau'') \in \phi(w'')$.

► Now using $\gamma_2 = \gamma' \wedge_{k_1}^{k_2} \gamma_1$ and $\tilde{\gamma}|_{[\tilde{k}-l, \tilde{k}-1]} = \gamma_1|_{[k_1-l, k_1-1]}$ from above gives $\gamma'' = \tilde{\gamma} \wedge_{k_2}^{\tilde{k}} \gamma_2 = \tilde{\gamma} \wedge_{k_2}^{\tilde{k}} (\gamma' \wedge_{k_1}^{k_2} \gamma_1) = \tilde{\gamma} \wedge_{k_1}^{\tilde{k}} \gamma_1 = \tilde{\gamma} \wedge_{k_1-l}^{\tilde{k}-l} \gamma_1$ implying

$$\forall \hat{k} \geq \tilde{k} . \gamma''|_{[\hat{k}-l, \hat{k}-1]} = \gamma_1|_{[\hat{k}-\tilde{k}+k_1-l, \hat{k}-\tilde{k}+k_1-1]} \quad (19)$$

► Remember that we have to show

$$\forall k \geq k_2 . \left[\begin{array}{l} \exists t'_2 \in \tau_2^{-1}(k), t'_2 > t_2 . \\ \quad \left[\begin{array}{l} (z_1(k - k_2 + k_1), x_2(t'_2)) \in \mathcal{R}_l^{-1} \end{array} \right] \end{array} \right]$$

Using $\hat{k} := k - k_2 + \tilde{k}, \mathcal{R}_l^{-1}$ from (16) and $(\gamma_1, z_1) \in \mathcal{B}_S^{\uparrow}$ this is equivalent to

$$\forall \hat{k} \geq \tilde{k} . \left[\begin{array}{l} \exists t'_2 \in \tau_2^{-1}(\hat{k} - \tilde{k} + k_2), t'_2 > t_2 . \\ \quad \left[\begin{array}{l} x_2(t'_2) \in \mathcal{X}_E(\gamma_1|_{[\hat{k}-\tilde{k}+k_1-l, \hat{k}-\tilde{k}+k_1-1]}), \end{array} \right] \end{array} \right]$$

and using $\tau''^{-1} = \tilde{\tau}^{-1} \wedge_{k_2}^{\tilde{k}} (\tau_2^{-1} + \tilde{t} - t_2), x'' = \tilde{x} \wedge_{t_2}^{\tilde{t}} x_2$, (19) and $\hat{t} := t'_2 + \tilde{t} - t_2$ it is equivalent to

$$\forall \hat{k} \geq \tilde{k} . \exists \hat{t} \in \tau''^{-1}(\hat{k}), \hat{t} > \tilde{t} . x''(\hat{t}) \in \mathcal{X}_E(\gamma''|_{[\hat{k}-l, \hat{k}-1]}).$$

Now observe that the last statement is true from (9), since $(w'', x'') \in \mathcal{B}_S$ and $(\gamma'', \tau'') \in \phi(w'')$ from above, what proves the statement. ■

Proof of Lemma 6 “ \Leftarrow ”:

1.) Show $\mathcal{B}_E = \mathcal{B}_E^{\uparrow}$:

• Observe that $\mathcal{B}_E \subseteq \mathcal{B}_E^{\uparrow}$ by definition and $\mathcal{R}_l^{-1} \in \mathfrak{R}_l(\Sigma_S^{\phi, \uparrow}, \Sigma_S^\phi)$ implies $\mathcal{B}_E|_{[l, \infty)} = \mathcal{B}_E^{\uparrow}|_{[l, \infty)}$.

• Show $\mathcal{B}_E^{\uparrow}|_{[0, l-1]} \subseteq \mathcal{B}_E|_{[0, l-1]}$:

► Observe that Lemma 2 implies $\mathcal{B}_E^{\uparrow}|_{[0, l-1]} = Z_E^l$.

► Now (16) and $\mathcal{R}_l^{-1} \in \mathfrak{R}_l(\Sigma_S^{\phi, \uparrow}, \Sigma_S^\phi)$ implies $\forall \zeta \in Z_E^l . \exists \xi \in X_E^l . \xi \in \mathcal{X}_E(\zeta)$.

► With Def. 3, this implies $\forall \zeta \in Z_E^l . \exists \gamma \in \mathcal{B}_E . \gamma|_{[0, l-1]} = \zeta$ what proves the statement.

2.) Show (12a) holds for $\mathcal{R}_{\mathcal{X}}$: by construction.

3.) Show (11b) holds for $\mathcal{R}_{\mathcal{X}}$:

• Pick the signals \cdot_1 and \cdot' as in the first part, i.e. “ \Rightarrow ”, of the proof and observe, that $(x_1(t_1), x'(t_2)) \in \mathcal{R}_{\mathcal{X}}$ and $(z_1(k_1), x'(t_2)) \in \mathcal{R}_l^{-1}$ as before.

• Since $\mathcal{R}_l^{-1} \in \mathfrak{R}_l(\Sigma_S^{\phi, \uparrow}, \Sigma_S^\phi)$ we can use (11b) and pick signals $(w_2, x_2) \in \mathcal{B}_S, (\gamma_2, \tau_2) \in \phi(w_2)$ s.t.

$$\left(\begin{array}{l} \gamma_2 = \gamma' \wedge_{k_1}^{k_2} \gamma_1 \\ \wedge \forall t \in T_2, t < t_2 . \left(\begin{array}{l} w_2(t) = w'(t) \\ \wedge x_2(t) = x'(t) \\ \wedge \tau_2(t) = \tau'(t) \end{array} \right) \\ \wedge x_2(t_2) = x'(t_2) \\ \wedge \forall k \geq k_2 . \left[\begin{array}{l} \exists t'_2 \in \tau_2^{-1}(k), t'_2 > t_2 . \\ \quad \left[\begin{array}{l} (z_1(k - k_2 + k_1), x_2(t'_2)) \in \mathcal{R}_l^{-1} \end{array} \right] \end{array} \right] \end{array} \right) \quad (20)$$

• Again, the first tree lines of (20) are equivalent to (18).

• To proof, that the last line of (18) also holds for this choice of signals \cdot_2 , remember that by construction $\forall k \in \mathbb{N}_0, t \in \tau_1^{-1}(k) . x_1(t) \in \mathcal{X}_E(z_1(k))$. With the parametrization of k in the last line of (18) and the last line of (20) we therefore have

$$\forall k \geq k_2, t'_1 \in \tau_1^{-1}(k - k_2 + k_1), t'_1 > t_1 . \left[\begin{array}{l} \exists t'_2 \in \tau_2^{-1}(k), t'_2 > t_2 . \\ \quad \left[\begin{array}{l} (x_1(t'_1), x_2(t'_2)) \in \mathcal{X}_E(z_1(k - k_2 + k_1)) \end{array} \right] \end{array} \right]$$

what proves the statement (from (17)). ■

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